Vol. XXII, No. 2; Nov.-Dec., 1948.

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to the managing editor, special papers to the editors of the departments for which they are intended, and general papers to some member of the Executive Committee.

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The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are \$3.00, sponsoring subscriptions \$10.00, single copies 65¢.

Subscriptions and related correspondence should be sent to: Inez James, 14068 Van Nuys Blvd., Pacoima, Calif.

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Entered as second-class matter March 23, 1948 at the post office Pacoima, California under act of congress of March 8, 1870.

CONTENTS

SOLUTIONS OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS AS ENTIRE ANALYTIC FUNCTIONALS OF THE COEFFICIENT FUNCTIONS

Aristotle D. Michal

Consider the linear differential system

$$(1) \qquad \frac{dw^i(x)}{dx} = a^i_j(x)w^j(x), \quad w^i(a) = w^i_0$$

in the n unknown functions $w'(x), \cdots, w^n(x)$ with the summation convention in operation. We shall assume that the n^2 functions $a^i_j(x)$ are continuous in the interval $a \leq x \leq b$.

We can replace the system (1) by the equivalent matric differential system

$$(2) \qquad \begin{cases} \dfrac{dw(x)}{dx} = A(x)w(x), \quad (A(x) \text{ continuous in } a \leq x \leq b), \\ w(a) = w_0 . \end{cases}$$

The unique solution of (2) is given by

$$(3) \qquad w(x) = \Omega^x_a[A(s)]w_0,$$

where $\Omega^x_t[A(s)]$ is the matrizant functional of $A(s)$ defined by

$$(4) \qquad \Omega^x_t[A(s)] = I + \int_t^x A(s_1)ds_1 + \int_t^x A(s_1)ds_1 \int_t^{s_1} A(s_2)ds_2 + \cdots + \cdots ,$$

where I is the unit matrix and $a \leq x, \ t \leq b$. To show the dependence of the solution $w(x)$ on $A(s)$, we shall write (3) as

$$(5) \qquad w[A(s)/x] = \Omega^x_a[A(s)]w_0 .$$

The Fréchet differentials and the expansions in infinite series will be considered in the Banach spaces B_1, B_2 and B_3. The elements of B_1 and B_2 are n-rowed square matrices of real functions of two real variables x, s and one real variable s respectively for $a \leq x, \ s \leq b$ and $a \leq s \leq b$. The elements of B_3 are column matrices of n real functions of one real variable x for $a \leq x \leq b$. The norm of B_1 will be

$$\|F\| = \max_{a \leq x, \ s \leq b} \|F(x,s)\|_n$$

while that of B_2 will be

$$\|A\| = \max_{a \leq s \leq b} \|A(s)\|_n$$

in terms of $\| \ \|_n$, the norm of the Banach space of all square matrices with n rows — any one of the equivalent Banach norms for square matrices can be used. Similarly the norm of B_3 will be

$$\|w\| = \max_{a \leq x \leq b} \|w(x)\|_c$$

in terms of $\| \ \|_c$, the norm of the Banach space of all column matrices with n elements — any one of the equivalent Banach norms for column matrices can be used. The spaces B_1 and B_2 are in fact *complete normed linear rings* whose unit elements correspond to the n-rowed unit matrix.

It is easy to show that the matrizant functional $\Omega_s^x[A(r)]$ is an entire analytic functional on B_2 to B_1 while $w[A(s)/x]$ is an entire analytic functional on B_2 to B_3.

We shall prove the following Theorem 1. In my forthcoming Acta Mathematica paper "On A Non-Linear Total Differential Equation in Normed Linear Spaces", the results of this theorem were included in §6 of that paper as an *instance* of a more general theory. Here we prove it directly by different methods without the aid of the general theory. It is evident of course from its definition (4) that Ω_t^x satisfies the following differential system as a function of x for any given t in $a \leq t \leq b$:

$$\frac{d\Omega_t^x}{dx} = A(x)\Omega_t^x, \quad \Omega_t^t = I.$$

THEOREM 1. *The matrizant functional $\Omega_r^t[A(s)]$ as a function on the space B_2 to B_1 satisfies the completely integrable non-linear differential system in Fréchet differentials*

$$(6) \qquad \delta\Omega_r^t[A(s)] = \int_r^t \Omega_s^t[A]\,\delta A(s)\,\Omega_r^s[A]\,ds, \quad \Omega_r^t[0] = I.$$

PROOF. That (6) is completely integrable follows from a Fréchet differentiation of (6) and elimination of first Fréchet differentials by means of (6). In fact, we have

$$(7) \qquad \delta_2\delta_1\Omega_r^t[A] = \int_r^t \left\{\int_s^t \Omega_p^t[A]\,\delta_2 A(p)\,\Omega_s^p[A]\,dp\right\}\delta_1 A(s)\,\Omega_r^s[A]\,ds$$
$$+\int_r^t \Omega_s^t[A]\,\delta_1 A(s)\,ds \int_r^s \Omega_p^s[A]\,\delta_2 A(p)\,\Omega_r^p[A]\,dp \quad (a \leq r,\ t \leq b).$$

But (7) is completely symmetric in $\delta_1 A(s)$ and $\delta_2 A(s)$ for *arbitrary* coefficients $\Omega_s^t[A]$. Hence $\delta_2\delta_1\Omega_r^t[A] = \delta_1\delta_2\Omega_r^t[A]$ as calculated above, is an *identity* in the independent variables Ω_r^t, $\delta_1 A(s)$ and $\delta_2 A(s)$. But this is what is meant by saying that the differential equation in (6) is completely integrable throughout the space.

From the unicity of the continuous solutions (5) of (2) on the one hand and of the matric Volterra integral equation with matric kernel $K(x,s) = -A(s)$ on the other, it is readily proved that the matric resolvent kernel $k(x,s)$ of $K(x,s)$ is related to the matrizant functional by the relation

$$(8) \qquad \Omega_s^x[A(p)] = I + \int_s^x k(x,r)\,dr.$$

Now $k(x,r)$ is a functional of $K(x,s)$, and $K(x,s)$ is a linear functional

of $A(s)$. Hence from the well known theorems on Fréchet differentials and Theorem 1 of a previous paper* (Michal, Proc. Nat. Acad. Sci. U.S.A., Aug. 1945) we find on using (8) that the first Fréchet differential of $\Omega_s^x[A(p)]$ exists and is given by

$$\text{(9)} \qquad \delta\Omega_s^x[A] = \int_s^x \Big[\delta A(r) + \int_r^x k(x,t)\delta A(r)dt + \int_r^x \delta A(t)k(t,r)dt + \int_r^x k(x,t)dt \int_r^t \delta A(s)k(s,r)ds\Big] dr.$$

If we invert the order of integration (Dirichlet's lemma is clearly valid on the basis of our hypotheses) in the last term in (9) and make a double use of (8) we find

$$\delta\Omega_s^x[A] = \int_s^x \Big\{\Omega_r^x[A]\delta A(r) + \int_r^x \Omega_t^x[A]\delta A(t)k(t,r)dt\Big\} dr.$$

Again, if we invert the order of integration in the second term on the right hand side and use (8), we obtain

$$\delta\Omega_s^x[A] = \int_s^x \Omega_r^x[A]\delta A(r)\Omega_s^r[A]dr.$$

That $\Omega_s^x[0] = I$ is obvious. Q. E. D.

THEOREM 2. *The differential system in Fréchet differentials*

$$\text{(10)} \qquad \begin{cases} \delta w[A(s)/x] = \int_a^x \Omega_s^x[A]\delta A(s)w[A/s]ds \\ w[0/x] = w_0 \qquad (a \le x,\ s \le b) \end{cases}$$

has a unique solution under our standing restrictions of continuity of $a_j^i(x)$ *in* $a \le x \le b$ *and definitions of Banach spaces. It is given by* (5), *the unique solution of the ordinary matric differential equation* (2) *when considered as a functional of* $A(x)$.

PROOF. That (5) satisfies the differential system (10) follows readily from (6) of the previous theorem. To prove that (5) is the unique solution of (10), let us assume that there does exist another solution

$$\text{(11)} \qquad y[A/x] \neq w[A/x]$$

of (10). Define $Z[A/x] = y[A/x] - w[A/x]$. Then clearly $Z[A/x]$ satisfies the following differential system

$$\text{(12)} \qquad \begin{cases} \delta Z[A(s)/x] = \int_a^x \Omega_s^x[A]\delta A(s)Z[A/s]ds \\ Z[0/x] = 0 \qquad (a \le x \le b). \end{cases}$$

*We merely use the fact that k satisfies formula (16) in that theorem.

Let us calculate the Fréchet differential

$$(13)\qquad \delta\left[Z[A/x] - \int_a^x A(s)Z[A/s]\,ds\right]$$

when $Z[A/x]$ is a solution of (12). On expanding out (13), using (12) and reversing the order of integration in the repeated integral, we obtain

$$(14)\qquad \delta\left[Z[A/x] - \int_a^x A(s)Z[A/s]\,ds\right]$$
$$= \int_a^x \left\{\Omega_s^x[A] - I - \int_s^x A(r)\Omega_s^r[A]\,dr\right\}\delta A(s)w[A/s]\,ds.$$

But the following functional identity is an easily shown property of the matrizant functional:

$$(15)\qquad \Omega_s^x[A] = I + \int_s^x A(r)\Omega_s^r[A]\,dr.$$

Hence

$$\delta\left[Z[A/x] - \int_a^x A(s)Z[A/s]\,ds\right] = 0$$

so that, since $Z[0/x] = 0$, we have

$$(16)\qquad Z[A/x] - \int_a^x A(s)Z[A/s]\,ds = 0.$$

But the unique continuous solution of this matric homogeneous Volterra integral equation is $Z[A/x] = 0$ for all x in $a \le x \le b$. This is absurd, and hence $w[A/x]$ is unique. Q. E. D.

It is clear from (6) that the matrizant functional $\Omega_r^t[A(s)]$ has successive Fréchet differentials of all orders. We shall prove the following theorem.

THEOREM 3*. *The nth successive Fréchet differential of the matrizant functional $\Omega_r^t[A(s)]$ exists for any positive integer n and any $A(s) \in B_2$, and the formula for the nth successive Fréchet differential with equal increments of $\Omega_r^t[A(s)]$ is given by*

$$(17)\qquad \delta^n\Omega_r^t[A] = n!\int_r^t \Omega_{s_1}^t \delta A(s_1)\,ds_1 \int_r^{s_1} \Omega_{s_2}^{s_1}\delta A(s_2)\,ds_2$$
$$\cdots \int_r^{s_{n-1}} \Omega_{s_n}^{s_{n-1}} \delta A(s_n)\Omega_r^{s_n}\,ds_n \qquad (n = 1,2,3,\cdots,\cdots).$$

PROOF. Let us write (6) in the following obvious manner

$$(18)\qquad \delta\Omega[A] = T(\Omega,\delta A,\Omega),$$

where T is a trilinear function on $B_1B_2B_1$ to B_1. We shall prove by

*This theorem was proved in Acta Mathematica paper, see §6 of that paper.

induction that the following result holds:

$$\delta^i\Omega[A] = i!T^i(\Omega[A],\delta A,\Omega[A]), \tag{19}$$

where $T^i(\Omega,\delta A,\Omega)$ is the ith iteration of the linear function $T(\Omega,\delta A,\Phi)$ of Φ evaluated for $\Phi=\Omega$. Obviously (19) holds for $i=1$. Assume it holds for $i=1,2,\cdots,n$.

Now

$$\delta^{n+1}\Omega[A] = n!\,\delta T^n(\Omega,\delta A,\Omega) \tag{20}$$

and

$$\delta T^n(\Omega,\delta A,\Omega) = \delta T\left[\Omega,\delta A,T^{n-1}(\Omega,\delta A,\Omega)\right].$$

Hence

$$\delta T^n(\Omega,\delta A,\Omega) = T\left[T(\Omega,\delta A,\Omega),\delta A,T^{n-1}(\Omega,\delta A,\Omega)\right] + T\left[\Omega,\delta A,\delta T^{n-1}(\Omega,\delta A,\Omega)\right] \tag{21}$$

On using (19) for $i=n-1,n$ we see that

$$\delta T^{n-1}(\Omega,\delta A,\Omega) = nT^n(\Omega,\delta A,\Omega). \tag{22}$$

Using (22) in (21), we obtain

$$\delta T^n(\Omega,\delta A,\Omega) = T\left[T(\Omega,\delta A,\Omega),\delta A,T^{n-1}(\Omega,\delta A,\Omega)\right] + nT^{n+1}(\Omega,\delta A,\Omega). \tag{23}$$

It can be shown with the aid of Dirichlet's lemma that

$$T\left[T(\Omega,\delta A,\Omega),\delta A,T^{n-1}(\Omega,\delta A,\Omega)\right] = T\left[\Omega,\delta A,T^n(\Omega,\delta A,\Omega)\right]. \tag{24}$$

Hence

$$\delta T^n(\Omega,\delta A,\Omega) = (n+1)T^{n+1}(\Omega,\delta A,\Omega). \tag{25}$$

If we use this result in (20), we find that (19) holds for $i=n+1$. The induction is complete and the proof of the theorem can be completed readily.

The following theorem follows without difficulty from (5) and Theorem 3.

THEOREM 4. *The solution $w[A/x]$ of the differential system* (2) *or* (10) *possesses successive Fréchet differentials of all orders. The nth successive differential with equal increments $\delta A(s)$ is given by*

$$\delta^n w[A/x] = n!\int_a^x \Omega_{s_1}^x[A]\,\delta A(s_1)\,ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}[A]\,ds_2 \cdots \int_a^{s_{n-1}}\Omega_{s_n}^{s_{n-1}}[A]\,\delta A(s_n)\,w[A/s_n]\,ds_n \quad (n=1,2,\cdots), \tag{26}$$

or equivalently by

$$\delta^n w[A/x] = n!\int_a^x \overset{*}{F}{}^n[A,\delta A/x,s]\,w[A/s]\,ds, \tag{27}$$

where $\overset{*}{F}{}^n[A, \delta A/x, s]$ *stands for the nth combined matric and integral composition power of* $\Omega_s^x[A]\,\delta A(s)$ *with variable limits.*

We shall need the following theorem in the proof of the generalized Taylor's series expansion for $w[A/x]$.

THEOREM 5. *Under our standing hypotheses, the matrizant functional* $\Omega_s^x[A(s)]$ *as a functional on* B_2 *to* B_1 *has a generalized Taylor's series expansion in successive Fréchet differentials valid for all* $\delta A(s)$ *in* B_2 *and for any chosen* $A_0(s)$ *in* B_2:

$$(28)\qquad \Omega_s^x[A_0(r) + \delta A(r)] = \Omega_s^x[A_0(r)] + \sum_{n=1}^{\infty} \frac{1}{n!}\,\delta^n\Omega_s^x[A_0(r)].$$

Furthermore, $\Omega_s^x[A_0(r) + \delta A(r)]$ *is an entire analytic functional of* $\delta A(r)$, *on* B_2 *to* B_1, *i.e.,* $\sum_{n=0}^{\infty} \mu_n\lambda^n$ *is an entire function of the numerical variable* λ *where* μ_n *is the modulus of the homogeneous polynomial* $\frac{1}{n!}\delta^n\Omega_s^x[A_0(r)]$ *of degree n in* $\delta A(r)$.

PROOF. We can write the matrizant as

$$(29)\qquad \Omega(A) = \sum_{i=0}^{\infty} \Omega_i(A)$$

where $\Omega_i(A)$ is the homogeneous polynomial of degree i on B_2 to B_1:

$$(30)\qquad \int_t^x A(s_1)ds_1 \int_t^{s_1} A(s_2)ds_2 \int_t^{s_2} \cdots \int_t^{s_{i-1}} A(s_i)ds_i .$$

Clearly, the sum of $i!$ terms obtained by permuting $A_1(s), \cdots, A_i(s)$ in

$$(31)\qquad \int_t^x A_1(s_1)ds_1 \int_t^{s_1} A_2(s_2)ds_2 \int_t^{s_2} \cdots \int_t^{s_{i-1}} A_i(s_i)ds_i .$$

and then dividing by $i!$ will be the polar $w_i(A_1, \cdots, A_i)$ of the homogeneous polynomial $\Omega_i(A)$.

It is easily shown that the matrizant $\Omega(A)$ is an entire analytic function on B_2 to B_1, i.e., $\sum_{i=0}^{\infty} m(\Omega_i)\lambda^i$ is a numerical entire function of λ with the understanding that $m(\Omega_i)$ is the modulus of the homogeneous polynomial $\Omega_i(A)$. Since B_1 is a complete space, this of course implies that $\sum_{i}^{\infty} \Omega_i(A)$ converges to $\Omega(A)$, and absolutely ($\sum_{i=0}^{\infty} \|\Omega_i(A)\|$ converges), for all $A(r) \in B_2$.

Define $F(A)$ by

$$(32)\qquad F(A) = \Omega(A_0 + A) \text{ for any chosen } A_0 \in B_2.$$

Evidently

$$(33)\qquad F(A) = \sum_{i=0}^{\infty} \Omega_i(A_0 + A)$$

and the series on the right converges and represents $F(A)$ for all $A \in B_2$. Now

(34) $\Omega_i(A) = w_i(A, \cdots, A)$

and hence

(35) $\Omega_i(A_0 + A) = \sum_{j=0}^{i} \binom{i}{j} w_i(A_0, \cdots, A_0, \overbrace{A, \cdots, A}^{j}).$

Let us consider the expanded series for $\Omega(A_0 + A)$

(36) $\sum_{i=0}^{\infty} \Omega_i(A_0 + A)$

by writing on each row the terms of same degree in A

(37)
$$\begin{aligned} w_0 + w_1(A_0) + w_2(A_0, A_0) + \cdots + w_n(A_0, \cdots, A_0) + \cdots + \cdots \\ + w_1(A) + 2w_2(A_0, A) + \cdots + nw_n(A_0, \cdots, A_0, A) + \cdots + \cdots \\ + w_2(A, A) + + \cdots + \binom{n}{2} w_n(A_0, \cdots, A_0, A, A) + \cdots + \cdots \\ + \cdots + \cdots \\ + \cdots \end{aligned}$$

This series is dominated by the series

(38)
$$\begin{aligned} \|w_0\| + m_1\|A_0\| + m_2\|A_0\|^2 + \cdots + m_n\|A_0\|^n + \cdots + \cdots \\ m_1\|A\| + 2m_2\|A_0\|\,\|A\| + \cdots + nm_n\|A_0\|^{n-1}\|A\| \\ + m_2\|A\|^2 + \cdots + \binom{n}{2} m_n\|A_0\|^{n-2}\|A\|^2 + \cdots + \cdots \\ + \cdots + \cdots \\ + \cdots \end{aligned}$$

where m_n is the modulus of the polar $w_n(A_1, \cdots, A_n)$ of $\Omega_n(A)$. If we sum this series by columns we obtain the series

(39) $\sum_{n=0}^{\infty} m_n (\|A_0\| + \|A\|)^n.$

Now if M_n is the modulus of $\Omega_n(A)$, we know from a general result of R. S. Martin (Calif. Institute of Technology thesis, 1932) relating the modulus of a homogeneous polynomial and that of its polar that

(40) $1 \le \frac{m_n}{M_n} \le \frac{n^n}{n!}.$

Since

(41) $\frac{n^n}{n!} < e^n$

we see from (40) that

(42) $m_n < M_n e^n.$

Hence each term in the series (39) is less than the corresponding

term in the series

$$(43) \qquad \sum_{n=0}^{\infty} M_n \left(e\,\|A_\circ\| + e\,\|A\|\right)^n$$

But $\sum_{n=0}^{\infty} M_n \lambda^n$ converges for all finite λ, since $\Omega(A)$ is an entire analytic function of A. This implies that (39) converges for all $A \in B_2$. Thus the norms of elements of (37) summed by columns is a convergent series for all $A \in B_2$. Hence (37) converges absolutely and, as in the classical numerical case, by the completeness of the space B_2 we can sum (37) in any other manner, say by rows. Now if we sum (37) by columns, we obtain the expansion for $F(A) = \Omega(A_\circ + A)$

$$(44) \qquad \Omega_\circ + \Omega_1(A_\circ + A) + \Omega_2(A_\circ + A) + \cdots + \cdots$$

with the aid of (35). On the other hand, if we sum (37) by rows we obtain the following expansion in Fréchet differentials with equal increments

$$(45) \qquad \Omega(A_\circ) + \sum_{i=1}^{\infty} [\delta\Omega_i(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \sum_{i=2}^{\infty} \frac{1}{2!} [\delta^2\Omega_i(A)]_{\substack{A=A_\circ \\ \delta A = A}}$$
$$+ \cdots + \sum_{i=n}^{\infty} \frac{1}{n!} [\delta^n\Omega_i(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \cdots + \cdots$$

since

$$(46) \qquad \delta^j\Omega_i(A) = i(i-1)\cdots(i-j+1)w_i(A,\cdots,A,\overbrace{\delta A,\cdots,\delta A}^{j}) \qquad \text{if } j \leq i.$$

But

$$(47) \qquad \delta^j\Omega_i(A) = 0 \qquad \text{if } j > 0.$$

Hence (45) can be written as

$$(48) \qquad \Omega(A_\circ) + \sum_{i=0}^{\infty} [\delta\Omega_i(A)]_{A=A_\circ} + + \frac{1}{2!} \sum_{i=0}^{\infty} [\delta^2\Omega_i(A)]_{\substack{A=A_\circ \\ \delta A = A}}$$
$$+ \cdots + \frac{1}{n!} \sum_{i=0}^{\infty} [\delta^n\Omega_i(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \cdots + \cdots$$

Now from the differentiability theorem on power series in normed linear spaces (Michal, Duke Math. Journal, 1946) we have

$$(49) \qquad \delta^j\Omega(A) = \sum_{i=0}^{\infty} \delta^j\Omega_i(A)$$

for any $A, \delta A$, since $\Omega(A)$ is an entire analytic function of A. Hence (48) can be written as

$$(50) \qquad \Omega(A_\circ) + [\delta\Omega(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \frac{1}{2!} [\delta\ \Omega(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \cdots$$
$$+ \frac{1}{n!} [\delta^n\Omega(A)]_{\substack{A=A_\circ \\ \delta A = A}} + \cdots + \cdots \quad .$$

We have therefore established the expansion (28) for $\Omega(A_\circ + \delta A)$ valid all $\delta A \epsilon B_2$.

It is to be observed that the results from formula (32) to formula (50) hold good for any two Banach spaces B_1 and B_2 and for *any* entire analytic function $\Omega(A)$ on B_2 to B_1. Nowhere in the proofs have we made use of any other restrictions on B_1, B_2 and $\Omega(A)$. The results in the proof after formula (50) were obtained by explicit use of some of the properties of the trilinear function T. However, again the Banach spaces B_1 and B_2 can be any Banach spaces over which T has the required properties.

Now what is the modulus of $[\frac{1}{n!}\,\delta^n\Omega(A)]_{\substack{A=A_\circ\\ \delta A=A}}$ as an nth degree homogeneous polynomial in A? We have by Theorem 3

$$[\frac{1}{n!}\,\delta^n\Omega(A)]_{\substack{A=A_\circ\\ \delta A=A}} = T^n\left[\Omega(A_\circ), A, \Omega(A_\circ)\right].$$

Hence

$$\left\| [\frac{1}{n!}\,\delta^n\Omega(A)]_{\substack{A=A_\circ\\ \delta A=A}} \right\| \leq \frac{\|\Omega(A_\circ)\|\, M^n\left[\Omega(A_\circ)\right]}{n!}\,\|A\|^n,$$

$$M\left[\Omega(A_\circ)\right] = (b-a)\,\|\Omega(A_\circ)\| .$$

Hence the modulus μ_n of the nth degree homogeneous polynomial $[\frac{1}{n!}\delta^n\Omega(A)]_{\substack{A=A_\circ\\ \delta A=A}}$ in A satisfies the inequality

$$\mu_n \leq \frac{(b-a)^n}{n!}\,\|\Omega(A_\circ)\|^{n+1}$$

so that $\sum_{n=0}^{\infty} \mu_n\lambda^n$ is an entire function of λ that is dominated term by term by $\|\Omega(A_\circ)\|\, e^{(b-a)\,\|\Omega(A_\circ)\|\lambda}$. Consequently the expansion in (28) defines an entire analytic functional of $\delta A(r)$ for all $\delta A(r) \epsilon B_2$. This completes the proof of Theorem 5.

With the aid of Theorem 3 we deduce immediately the following important corollary yielding a new expansion for the matrizant functional.

COROLLARY. *The matrizant functional $\Omega_r^t[A(s)+B(s)]$ is, for any $A(s) \epsilon B_2$, an entire analytic functional of $B(s)$ — considered on B_2 to B_1 — and has the following expansion in terms of $\Omega_r^t[A(s)]$ and $B(s)$:*

$$(51)\qquad \Omega_r^t[A(s)+B(s)] = \Omega_r^t[A(s)] + \sum_{n=1}^{\infty} \int_r^t \Omega_{s_1}^t[A]B(s_1)ds_1 \int_r^{s_1} \Omega_{s_2}^{s_1}[A]B(s_2)ds_2 \cdots \int_r^{s_{n-1}} \Omega_{s_n}^{s_{n-1}}[A]B(s_n)\Omega_r^{s_n}[A]\,ds_n .$$

We can now state the following theorem and its two corollaries. They follow from Theorem 2, Theorem 4 and Theorem 5 and its corollary.

THEOREM 6. *Under our standing hypotheses, the unique solution $w[A(s)/x]$ of system* (10) *or* (2) *as a functional on B_2 to B_2 has a generalized Taylor's series expansion in successive Fréchet differentials valid for all $\delta A(s)$ in B_2 and for any chosen $A_0(s)$ in B_2:*

$$(52)\qquad w[A_0(s)+\delta A(s)/x] = w[A_0(s)/x] + \sum_{n=1}^{\infty}\frac{1}{n!}\delta^n w[A_0(s)/x].$$

In fact $w[A_0(s)+\delta A(s)/x]$ is an entire analytic functional of $\delta A(s)$ — on B_2 to B_3 — so that $\sum_{n=0}^{\infty} v_n\lambda^n$ is an entire analytic function of the numerical vairable λ and v_n is the modulus of the homogeneous polynomial $\frac{1}{n!}\delta^n w[A_0(s)/x]$ of degree n in $\delta A(s)$.

COROLLARY 1. *The functional $w[A(s)+B(s)/x]$ is an entire analytic functional of $B(s)$ — considered on B_2 to B_3 — for any chosen $A(s)$ in B_2. It has the following expansion in terms of $w[A(s)/x]$ and $B(x)$ as an entire analytic functional of $B(s)$:*

$$(53)\qquad w[A(s)+B(s)/x] = w[A(s)/x] + \sum_{n=1}^{\infty}\int_a^x \Omega_{s_1}^x[A]B(s_1)ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}[A]B(s_2)ds_2 \cdots \int_a^{s_{n-1}}\Omega_{s_n}^{s_{n-1}}[A]B(s_n)w[A/s_n]ds_n.$$

COROLLARY 2. *Let K be the modulus of the bilinear function Fw on B_1B_3 to B_3. Then the following inequality holds for each element of the column matrix on the left and for each x in $a\le x\le b$*

$$(54)\qquad w[A(r)+B(r)/x] - w[A(r)/x] - \sum_{n=1}^{i}\int_a^x\Omega_{s_1}^x[A]B(s_1)ds_1\int_a^{s_1}\Omega_{s_2}^{s_1}[A]B(s_2)ds_2 \cdots \int_a^{s_{n-1}}\Omega_{s_n}^{s_{n-1}}[A]B(s_n)w[A/s_n]ds_n$$
$$\le K\,\|w_0\|\,\|\Omega[A]\|\sum_{n=i+1}^{\infty}\frac{(b-a)^n\,\|\Omega[A]\|^n\,\|B\|^n}{n!}.$$

It is to be observed that the n inequalities (54) give a measure of the error commited in stopping with the differential correction of the ith order, i.e., with the term $n=i$ in (53), an expansion which defines the exact solution of the differential system

$$(55)\qquad \frac{dw^i(x)}{dx} = [a_j^i(x)+b_j^i(x)]w^j(x),\quad w^i(a)=w_0^i$$

in terms of the solution and coefficients of (1) and of the n^2 arbitrary continuous perturbation functions $b_j^i(x)$. The inequalities (54) are obtained from similar inequalities for the matrizant functional.

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SINGULAR MEASURABLE SETS AND LINEAR FUNCTIONALS

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1. *Introduction.* By making use of a property of singular sets that is stronger than that given in [2][1] or in [6] and known theorems for linear topological spaces, for example, Theorem 3 of [4], we obtain an extension of a theorem by Arens [1] on the existence of a multiplicative-additive continuous functional on the space L^{ω}, where the integral is over an abstract space and the measure of the space may be infinite. The method of proof is then applied to obtain a direct and simple proof of a theorem due to Day [2] on the existence of linear functionals on the space L^p, $0<p<1$. The theorems and definitions on integration needed for this paper are contained in Chapter I of [5]. For a more complete discussion of the spaces L^{ω} and L^p, $0<p<1$, and for the motivation for considering this type of problem see [1] and [2].

$\mathfrak{X}$ denotes an additive class of sets in an abstract space X. μ is a measure $(\mathfrak{X})$. $L^p_\mu(X)$, $p>0$, denotes the space of real-valued measurable $(\mathfrak{X})$ functions f for which $\|f\|_p = (\int_X |f|^p d\mu)^{1/p} < \infty$. It is sufficient for this paper to point out that $L^p_\mu(X)$ is a pseudo-normed linear space with pseudo-norm[2] $\|f\|_p$. For $p \geq 1$, $\|f\|_p$ is a norm and $L^p_\mu(X)$ is a normed, complete topological linear algebra. For $0<p<1$, $\|f\|_p$ does not satisfy the triangular inequality but the weaker inequality $\|f+g\|_p \leq 2^{1/p-1}[\|f\|_p + \|g\|_p]$, and in this case $L^p_\mu(X)$ is a pseudo-normed, complete topological linear algebra.

The space $L^{\omega}_\mu(X)$ is the set of those functions f such that $f \in L^p_\mu(X)$ for all $p \geq 1$. If X is of finite measure, then given $p \geq 1$, $\|f\|_p \leq \mu^{1/r}(X)\,\|f\|_q$, $\frac{1}{q}+\frac{1}{r}=\frac{1}{p}$, and $n(f,\alpha) = a\|f\|_p\,\mu^{1/p}(X)$, where $\alpha = (a,p)$, is a pseudo-norm with respect to $R^{+2} = [(a,p);\ a>0,\ p>0]$. $(a_1, p_1) \geq (a_2, p_2)$ if $p_1 \geq p_2$ and $a_1 \geq a_2$. $L^{\omega}_\mu(X)$ is a convex, pseudo-normed, complete topological linear algebra. If $\mu(X)$ is not finite, then the topology of $L^{\omega}_\mu(X)$ can no longer be generated by $\|f\|_p$. The sets $[f;\ a\|f\|_p < 1]$ do not in general form a complete neighborhood system. For example, let X be the interval $(0,\infty)$, $f = e^{-ax}$, $a>0$, and μ the Lebesgue measure. If $p_1 \neq p_2$, it cannot be that for all $a>0$ and some $K>0$ that $\|e^{-ax}\|_{p_1} \leq K\|e^{-ax}\|_{p_2}$. However, the topology of $L^{\omega}_\mu(X)$ can

(1) The numbers in brackets refer to the Bibliography at the end of the paper.

(2) See [3] or [4]. L^p is therefore a linear topological space with the topology generated by $\|f\|_p$. f and g are considered to be equal, $f=g$, if $f(x)=g(x)$ a.e. on X.

be generalized in the following manner. Let $\mathfrak{F}$ be the class of sets in $\mathfrak{X}$ which are of finite measure. Then for $\alpha = (a,p,F) \in R^+R^+\mathfrak{F}$ define $n(f,\alpha) = a\mu^{1/p}(F)\,||C_F\cdot f||_p$ where C_F is the characteristic function of F. $C_F(x) = 1$ for $x \in F$ and $C_F(x) = 0$ otherwise. Define $\alpha_1 > \alpha_2$, if $F_2 \subset F_1$, $p_1 \geq p_2$ and $a_1 \geq a_2$. $n(f,\alpha)$ is a pseudo-norm and as before $L^{\omega}_{\mu}(X)$ is a convex, pseudo-normed, complete topological linear algebra with the neighborhoods of the zero element defined by $[f;\ n(f,\alpha) < 1]$. This latter neighborhood system is equivalent to the former neighborhood system for $\mu(X) < \infty$, i.e., when $\mathfrak{F} = \mathfrak{X}$.

2. Saks [6] defines a set $E \in \mathfrak{X}$ to be *singular* if $0 < \mu(E) < \infty$ and if for every $E' \in \mathfrak{X}$ either $\mu(E\cdot E') = 0$ or $\mu(E-E') = 0$. $\mathfrak{X}$ is said to be *singular* with respect to μ if $\mathfrak{X}$ contains a singular set. In [2] it is shown that for a large class of $\mathfrak{X}$ and μ, $\mathfrak{X}$ is not singular. An example of a space $L^p_{\mu}(X)$ in which $\mathfrak{X}$ is singular is the space l^p.(3) Saks [6] has shown that *$\mathfrak{X}$ is not singular if and only if given $\epsilon > 0$ and $E \in \mathfrak{X}$, $0 < \mu(E) < \infty$, there are a finite number of sets $E_i \in \mathfrak{X}$, $i = 1, \cdots m$ with the property that $E = \sum_{i=1}^{m} E_i$ and $\mu(E_i) < \epsilon$.* We wish to make use of the following stronger property:

LEMMA 1. *$\mathfrak{X}$ is not singular with respect to μ if and only if given $E \in \mathfrak{X}$ and $0 < \lambda < 1$ there is an $E' \in \mathfrak{X}$ such that $E' \subset E$ and $\mu(E') = \lambda\mu(E)$.*

Proof: The sufficiency is clearly true and the necessity is trivial for $\mu(E) = 0$ or $\mu(E) = \infty$. Assume that $\mathfrak{X}$ is not singular and that $0 < \mu(E) < \infty$. Then by Saks' theorem stated above, E can be covered by a finite number of measurable sets, $E = \sum_{i=1}^{m} E_i$, with $\mu(E_i) < \epsilon < 1$. Hence for some m_1, $F_1 = \sum_{i=1}^{m_1} E_i$, $0 \leq (F_1) - \lambda\mu(E) < \epsilon$. By subdividing E_{m_1}, we see that we can obtain an $F_2 \subset F_1 \subset E$ with $0 \leq \mu(F_1) - \lambda\mu(E) < \epsilon^2$, and using this procedure we see by induction that there exists a monotone sequence of sets $F_n \in \mathfrak{X}$ with $F_n \subset E$, $\lim \mu(F_n) = \lambda\mu(E)$. The desired E' is therefore $E' = \lim F_n$.

We shall also need the following construction.

LEMMA 2. *For $f \in L^p_{\mu}(\mathfrak{X})$, $p > 0$, there is a sequence $f_n \in L^p$, $n = 1, 2, \cdots$, such that:*

1. $|f_n(x)|^p < n$, $x \in X$

2. $f_n \to f$ *in* L^p (i.e. $||f - f\,||_p \to 0$) *as* $n \to \infty$.

(3) For l^p, X is the set of positive integers, $\mathfrak{X}$ is the class of all subsets of X, $\mu(E)$ is the number of elements, and the singular sets of $\mathfrak{X}$ are the sets which consist of a single element.

3. $\mu(F_n) < \infty$ *and* $[x;\ f(x) \neq 0] = \sum_{n=1}^{\infty} F_n$,

where $F_n = [x;\ f_n(x) \neq 0]$.

Proof: Define for $n = 1, 2, \cdots$,

$$E_{0,n} = [x;\ |f(x)|^p < \frac{1}{n}]$$

$$E_{1,n} = [x;\ |f(x)|^p > n]$$

$$F_n = [x;\ n \geq |f(x)|^p \geq \frac{1}{n}]$$

$$f_{i,n}(x) = \begin{cases} f(x) & , x \in E_{i,n} \\ 0 & , \text{otherwise.} \end{cases}$$

Now clearly $f = f_{0,n} + f_{1,n} + f_n$, $f_{0,n}$, $f_{1,n}$, and $f_n \in L^p$, and $\mu(F_n) < \infty$, for each n. We will show that $||f_{0,n}||_p \to 0$ and $||f_{1,n}||_p \to 0$ as $n \to \infty$, and hence that $||f - f_n|| \leq 2^{1/p} [\, ||f_{0,n}||_p + ||f_{1,n}||_p\,] \to 0$ as $n \to \infty$. Clearly $\lim f_{0,n}(x) = 0$ for each $x \in X$ and by Lebesque's theorem on term-by-term integration $||f_{0,n}|| \to 0$. Also $||f||_p = \int_X |f|^p \, d\mu \geq \int_{E_{1,n}} |f|^p \, d\mu \geq n\mu(E_{1,n})$. Hence $\mu(E_{1,n}) \to 0$. Since $||f_{1,n}||_p^p = \int_{E_{1,n}} |f|^p \, d\mu$, we have by the theorem of Radon-Nikodym that $||f_{1,n}||_p \to 0$. Hence f_n is the required sequence. 1 and 3 are true by definition.

THEOREM 1. *The following statements concerning L^ω are equivalent:*

1. $\mathfrak{X}$ *is singular.*

2. *There is a non-null linear functional $\phi(f)$ on L^ω with the property that $\phi(f \cdot g) = \phi(f) \cdot \phi(g)$, $f, g \in L^\omega$.*

3. *There is an open convex set $U \subset L^\omega$ which is properly contained in L^ω, contains the null function 0, the zero element of L^ω, and is such that $U \cdot U \subset U$.*

Proof: (1 implies 2): Let E be a singular set of $\mathfrak{X}$. Then $f(x) = \bar{f}_E$, a constant, almost everywhere on E for each $f \in L^p$. Define $\phi(f) = \bar{f}_E$. Now $\bar{f}_E \mu^{1/p}(E) = n(f, \alpha)$, $\alpha = (1, p, E)$, and $|\phi(f)| = n(f, \alpha)$. Hence $\phi(f)$ is continuous on L^ω. It is evident that $\phi(f + g) = \phi(f) + \phi(g)$ and that $\phi(f) \cdot \phi(g) = \phi(fg)$ for $f, g \in L^p$. ϕ is non-null since the characteristic function of E is in L^ω.

(2 implies 3): Let ϕ satisfy 2°, then $U = [f;\ |\phi(f)| < 1]$ satisfies 3. See Theorem 3 of [4].

(3 implies 1): Assume that 3° is true and let $f \in L^p$ be such that $f \bar{\epsilon} U$. Since U is open and contains 0, $f \neq 0$, and there exist $\alpha = (a,p,E) \in R^{+2}\ F$, ($p$ can be taken to be an integer) such that $[g;\ n(g,\alpha) < 1] \subset U$. Using Lemma 2 we see that for a sufficiently large n_0, $0 < \mu(F_{n_0}) < \infty$, $f - f_{n_0} \in U$. If X is not singular, then by Lemma 1, F_{n_0} can be expressed as the sum of s disjoint sets $G_i \in x$, $i = 1,\cdots s$, with $\mu(G_i) = 1/s\mu(F_{n_0})$. Define

$$g_0 = 2(f - f_{n_0}),\quad g_i(x) = \begin{cases} 2s\, f_{n_0}(x), & x \in G_i \\ 0 & , \text{otherwise} \end{cases},\quad i = 1,2,\cdots .$$

$n(g_i^{\frac{1}{2p}}, \alpha) = K\, \|g_i^{\frac{1}{2p}}\|_p \leq K(\frac{2}{s})^{\frac{1}{2p}}\ [n_0^{\frac{1}{2p}}\ (F_{n_0})]^{1/p}$. Hence for sufficiently large s, $g_i^{\frac{1}{2p}} \in U$, and therefore $g_i \subset U^{2p} \subset U$. But $f = \frac{1}{2} g_0 + \sum_{i=1}^{s} \frac{1}{2s} g_i$, and since U is convex, $f \in U$. This is a contradiction.

3. In the proof of theorem 1, we showed essentially that *if $f = 0$ almost everywhere on each singular set in $\mathfrak{X}$, then $\phi(f) = 0$ for every ϕ satisfying 2 of theorem* 1. Making use of this and the following two lemmas we will show that every functional on L^ω is of the form given in "1 implies 2" in theorem 1.

LEMMA 3. *Given $f \in L^\omega_\mu$ (X), there exists a countable sequence E_n (f), $n = 1,2,\cdots$, of disjoint, singular sets of $\mathfrak{X}$ with the property that each singular set E of $\mathfrak{X}$ for which $f \neq 0$ almost everywhere on E differs from one $E_n(f)$ by at most a set of measure zero.*

Proof: From 3 of Lemma 2 we see that the set $[x;\ f(x) \neq 0]$ can be expressed as the sum of a countable number of sets $F_n \in \mathfrak{X}$ where each F_n is of finite measure. From the nature of singular sets any singular set in $\mathfrak{X}$ with the property that $f(x) \neq 0$ almost everywhere on E must be almost everywhere contained in some F_n. Now in a set $E \in \mathfrak{X}$ of finite measure any aggregate of almost-everywhere disjoint subsets of positive measure is at most countable. Hence the number of almost-everywhere disjoint singular sets E which are such that $f(x) \neq 0$ almost everywhere on E and which are contained in one of the F_n must be at most countable. Since a countable sequence of countable sets is at most countable, it should be clear that the desired sequence $E_n(f)$ can be selected from the set of all E for which $f(x) \neq 0$ almost everywhere on E.

LEMMA 4. *If ϕ is a functional satisfying 2 of theorem 1, E a singular set of $\mathfrak{X}$ such that $\phi(C_E) \neq 0$, then $\phi(C_E) = 1$ and $\phi(f) = \mu^{-1}(E)\int_E f\, d\mu$ for each $f \in L^\omega$.*

Proof: Clearly $C_E \cdot C_E = C_E$ and $\phi^2(C_E) = \phi(C_E)$, which proves that $\phi(C_E) = 1$. Now $f(x) = \overline{f}_E$, a constant, almost everywhere on E, and $\overline{f}_E = \mu^{-1}(E) \int_E f\, d\mu$. Hence $\phi(f) = \phi(C_E) \cdot \phi(f) = \phi(C_E \cdot f)$ $\phi(\overline{f}_E \cdot C_E) = \overline{f}_E$, which completes the proof.

THEOREM 2. *If E is a singular set then $\phi(f) = \mu^{-1}(E) \int_E f\, d\mu$ satisfies 2 of theorem 2. Conversely every functional satisfying 2 of theorem 1 is of this form.*

Proof: The first part of the theorem is a restatement of what was shown in the proof of "1 → 2" of theorem 1. Let $g \in L^\omega$ be such that $\phi(g) \neq 0$. Let $E_n(g) = G_n$ be the sequence of Lemma 3, and define $h = \sum_n C_{G_n} \cdot g$. The limit of the sum exists, since for each $x \in \mathfrak{X}$ at most one term in the series is not zero. Also $\|h\|_p \leq \|f\|_p$ for each p and $h \in L^\omega$. Now by the remark made at the beginning of this section $\phi(g) = \phi(h)$. From the continuity of ϕ we then have that $\phi(g) = \phi(h) = \sum_n \phi(C_{G_n} \cdot g) = \sum_n \phi(C_{G_n}) \cdot \phi(g)$. Since $\phi(g) \neq 0$, it must be that for one G_n (and only one as a matter of fact), $\phi(C_{G_n}) \neq 0$. The application of Lemma 4 completes the proof of the theorem.

4. The theorem due to Day [2], Theorem 5, on the existence of linear functionals on the space $L^p_\mu(X)$, $0 < p < 1$, where $\mu(X)$ is not necessarily finite, can be obtained immediately by a slight modification in the proof of Theorem 1 above.

THEOREM 3. *If $f = 0$ almost everywhere on every singular set of $\mathfrak{X}$, then $\phi(f) = 0$ for each linear functional ϕ on $L^p_\mu(X)$, $0 < p < 1$.*

Proof: Assume that $f = 0$ a.e. on every singular set of $\mathfrak{X}$ and that $\phi(f) \neq 0$. Then $U = [g;\ g \in L^p,\ |\phi(g)| < \phi(f)]$ is an open convex set containing the origin and $f \bar{\epsilon} U$. so in "3 implies 1" in Theorem 1 we obtain $\|g_i\|_p \leq 2s^{1-1/p}(n_0\, \mu(F_{n_0}))^{1/p}$, since F_{n_0} cannot be a singular set. Hence for sufficiently large s $g_i \in U$ and $f = \frac{1}{2} g_0 + \sum_{i=1}^{s} \frac{1}{2s} g_i \in U$, since U is convex. This is a contradiction and hence $\phi(f) = 0$.

The representation of a linear functional on L^p, $0 < p < 1$, is obtained as in Theorem 2. $\phi(f) = \sum_n \overline{f}_{F_n} \phi(C_{F_n})$, where it is to be remembered that with each f there is associated a sequence of singular sets $F_n = E_n(f)$ as in Lemma 3.

BIBLIOGRAPHY

1. R. Arens, *The space L^{ω} and convex topological rings,* Bull. Amer. Math. Soc., vol. 52 (1946) pp. 931-935.

2. M. M. Day, *The spaces L^p,* $0<p<1$, Bull. Amer. Math. Soc., vol. 46 (1940), pp. 816-823.

3. D. H. Hyers, *Pseudo-normed linear spaces and Abelian groups,* vol. 5 (1939), pp. 628-634.

4. J. LaSalle, *Pseudo-normed linear spaces,* Duke Math. Jr., vol. 8 (1941), pp. 131.135.

5. S. Saks, *Theory of the Integral,* Warsaw, 1937.

6. ———, *Addition to the note on some functionals,* Trans. Amer. Math. Soc., vol. 35 (1933), pp. 965-970

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SOME NEW INTRINSIC PROPERTIES OF CUBICS AND QUARTICS

J. Russell Franks

INTRODUCTION

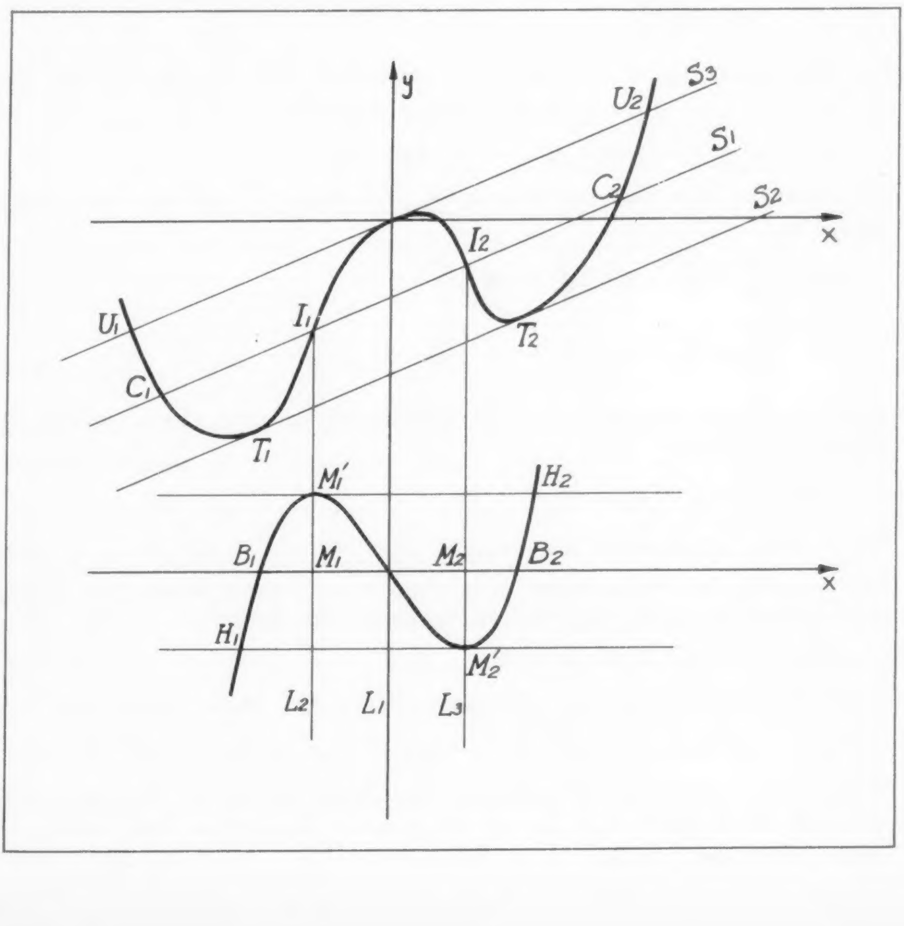
Consider the plane curve of the third degree and its integral curve, which after rigid motion in the plane reduce to:

(1) $y = a_0 x^3 + a_1 x^2 + a_2 x + a_3 \qquad (a_0 \neq 0)$

(2) $y = b_0 x^4 + b_1 x^3 + b_2 x^2 + b_3 x + b_4$

Two critical points on the cubic which correspond to the two inflexion points of the integral curve are shown to exist. Two other characteristic points on the cubic are defined. By means of these four points a ratio, invariant under rigid transformation, is calculated.

Similarly eight characteristic points on the quartic are located and analogous invariant ratios set up.



LOCATION OF THE CRITICAL POINTS M_1' AND M_2' OF THE CUBIC, INDEPENDENT OF THE AXES

For the present we shall assume that the integral curve of the cubic is given and the inflexion points I_1 and I_2 of this curve have real coordinates. Draw a chord connecting I_1 and I_2. Draw another parallel to it at a convenient distance from it, but intersecting the quartic at two (or four) points. Bisect both chords and draw a line (L_1) through the ~~two~~ midpoints (see the graph). Draw L_2 and L_3 through I_1 and I_2, both parallel to L_1.

Then L_1 passes through the inflexion point of the cubic and the lines L_2 and L_3 pass through its critical points M_1' and M_2'.

PROOF The translation $x = \dfrac{-a_1}{3a_0} = \dfrac{-b_1}{4b_0}$ reduces (2) to the form

(3) $y = b_0\ [x^4 + 2px^2 + 4Qx] + R$

and (1) to

(4) $y = a_0\ [x^3 + px] + a_0 Q$

Equate the second derivative of (3) to zero, and there results

(5) $x = \pm\sqrt{\dfrac{-p}{3}}$

Let the equation of the secant through I_1 and I_2, be of the form,

(6) $y = b_0\ [Mx + b] + R$ whence from (3)

(7) $b_0\ [x^4 + 2px^2 + 4Qx] = b_0\ [Mx + b]$

Substituting first the negative value of x from (5) and then the positive value in (7) gives

(8) $\dfrac{p^2}{9} - \dfrac{2p^2}{3} - 4Q\sqrt{\dfrac{-p}{3}} = -M\sqrt{\dfrac{-p}{3}} + b$

(9) $\dfrac{p^2}{9} - \dfrac{2p^2}{3} + 4Q\sqrt{\dfrac{-p}{3}} = M\sqrt{\dfrac{-p}{3}} + b$

From these two equations $b = \dfrac{-5p^2}{9}$ and $M = 4Q$, so that the equation of the secant S_1 becomes

(10) $y = b_0\ [4Qx - \dfrac{5p^2}{9}] + R$

Since this secant cuts the quartic at $x = \pm\sqrt{\dfrac{-p}{3}}$, that is at I_1 and I_2, the midpoint of the segment $\overline{I_1 I_2}$ lies on L_1, the y axis. Any arbitrary line, parallel to S_1 will of course be of the form

(11) $y = b_0\ [4Qx + K] + R$

From (3) and (11) we get the quadratic in x^2 whose roots are

$$x = \pm\sqrt{-p - \sqrt{p^2 + K}} \qquad \text{and} \qquad x = \pm\sqrt{-p + \sqrt{p^2 + K}}$$

It is clear that the midpoint of the chord having as abscissae either pair of roots will also lie on the y axis. Therefore the line L_1 coincides with the y axis.

Now, the cubic (4) has the integral curve

(12) $y = b_0\ [x^4 + 2px^2 + 4Qx] + R$

where fractions have been cleared and $b_0 = \frac{a_0}{4}$. We set R equal to zero, since our considerations are independent of vertical translations.

The lines L_2 and L_3 pass through I_1 and I_2 of the quartic and hence through the cubic at $M_1' = -\sqrt{\frac{-p}{3}}$ and $M_2' = \sqrt{\frac{-p}{3}}$. These two points, marked M_1' and M_2', are now maximum and minimum points of the cubic with respect to any x axis perpendicular to the line L_1. Thus the lines upon which M_1' and M_2' lie are determined independently of the axes. We can now prove the following theorem:

THEOREM I *Draw a secant through the point of inflexion of the cubic (1) and perpendicular to a normal through either of the other two critical points. Then either segment of this secant subtended by the point of inflexion and either branch of the cubic, is to the segment subtended by the point of inflexion and either normal, as the square root of three is to one.*

PROOF For the cubic select the line through its point of contact with L_1 and perpendicular to L_1 as an x axis, then (4) reduces to

(13) $y = a_0\ [x^3 + px]$

This axis is perpendicular to the normals L_2 and L_3 through M_1' and M_2' and makes contact with them at M_1 and M_2. The equations of lines L_2 and L_3 are $x + \sqrt{\frac{-p}{3}} = 0$ and $x - \sqrt{\frac{-p}{3}} = 0$ so that $\overline{OM_1} = \overline{M_2O} = \sqrt{\frac{-p}{3}}$.

The cubic cuts the x axis at the two additional characteristic points B_1 and B_2 or at $x = \pm\sqrt{-p}$ as can be seen from (13) by setting y equal to zero.

We therefore have the ratio,

(14) $\dfrac{\overline{OB_1}}{\overline{OM_1}} = \dfrac{\overline{B_1B_2}}{\overline{M_1M_2}} = \dfrac{2\sqrt{-p}}{2\sqrt{\frac{-p}{3}}} = \sqrt{3}.$

Referring to the graph, the line S_1 has the equation (11) already developed; fitting it to the quartic (4) gives,

(15) $b_0\ [x^4 + 2px^2 + 4\ x] = b_0\ [4Qx - \frac{5p^2}{9}]$

a quadratic equation in x^2 whose roots are

(16) $C_1 = -\sqrt{\frac{-5p}{3}}$, $I_1 = -\sqrt{\frac{-p}{3}}$, $I_2 = \sqrt{\frac{-p}{3}}$ and $C_2 = \sqrt{\frac{-5p}{3}}$.

The line S_2 is tangent to the quartic at two distinct points. Its equation is

(17) $y = b_0\ [4Qx - p^2] + R,$

eliminating y between this and (4) gives

(18) $b_0\ [x^4 + 2px^2 + 4Qx] = b_0\ [4Qx - p^2]$

a perfect square in x^2. The double roots are

$$T_1 = -\sqrt{-p} \quad \text{and} \quad T_2 = \sqrt{-p}\,.$$

The line S_3 is tangent to the quartic at its point of contact with the line L_1 (here used as the origin). Its equation is

$$(19) \qquad y = b_0\ [4Qx] + R$$

Eliminating y between this equation and (3) gives

$$(20) \qquad x^2\ [x^2 + 2p] = 0$$

showing that (19) is tangent at the origin, as stated, and cuts the outside branches at $U_1 = -\sqrt{-2p}$ and $U_2 = \sqrt{-2p}$.

THEOREM II *Pass through the quartic,* $y = b_0\ [x^4 + 2px^2 + 4Qx]$ *the following lines* (S_1), $y = b_0\ [4Qx - \frac{5p^2}{9}]$, *(through the inflexion points)* (S_2), $y = b_0\ [4Qx - p^2]$, *(tangent at two distinct points)* (S_3), $y = b_0\ [4Qx]$, *(tangent at the origin); then the chords of the quartic formed by these three lines when taken in the proper order form three invariant ratios equal to the quadratic surds,* $\sqrt{2}$, $\sqrt{3}$ *and* $\sqrt{5}$.

PROOF The points of contact of the three lines have been defined above as the characteristic points T_1, T_2, U_1, U_2, C_1, C_2, I_1 and I_2. The ratios of the line segments determined by them are:

$$(21) \qquad \frac{\overline{U_1U_2}}{\overline{T_1T_2}} = \frac{2\sqrt{-2p}}{2\sqrt{-p}} = \sqrt{2}\,,$$

$$(22) \qquad \frac{\overline{T_1T_2}}{\overline{I_1I_2}} = \frac{2\sqrt{-p}}{2\sqrt{\frac{-p}{3}}} = \sqrt{3} \quad \text{and}$$

$$(23) \qquad \frac{\overline{C_1C_2}}{\overline{I_1I_2}} = \frac{2\sqrt{\frac{-5p}{3}}}{2\sqrt{\frac{-p}{3}}} = \sqrt{5}\,.$$

If p is greater than zero, the characteristic points of the cubic and its integral curve become imaginary (except for the inflexion point of the cubic and the point of tangency of the line $y = b_0\ [4qx]$ on the quartic). The algebra, however, carries over in toto and the invariant ratios still hold.

If p equals zero, the lines S_1, S_2 and S_3 coincide as do the lines L_1, L_2 and L_3. In this case the ratios become indeterminate and we define them as equal to those in the original two theorems which are the limits these ratios approach as p approaches zero.

University of California.

CURRENT PAPERS AND BOOKS

Edited by
H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

The purpose and policies of the first division of this department (Comments on Papers) derive directly from the major objective of the MATHEMATICS MAGAZINE which is to encourage research and the production of superior expository articles by providing the means for prompt publication.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited. Comments which express conclusions at variance with those of the paper under review should be submitted in duplicate. One copy will be sent to the author of the original article for rebuttal.

Communications intended for this department should be addressed to

H. V. Craig, Department of Applied Mathematics,
University of Texas, Austin 12, Texas.

Mathematics as a Culture Clue, and Other Essays. By Cassius Jackson Keyser. New York, Scripta Mathematics, Yeshiva University, 1947. vii + 277 pp. $3.75.

There are in all twelve essays in this volume of the proposed collected works of the late Professor Keyser. There is a fine portrait of Professor Keyser as frontespiece, and in the essay on Charles Sanders Peirce, three portraits of that pioneering logician and philosopher.

Some of the essays will be familiar to all who are interested in the philosophy of mathematics and in what Keyser called "the human worth of rigorous thinking". All are written in the author's characteristic style, happily recalling his personality. What the author says in introducing the first essay, *The Meaning of Mathematics,* applies with a few verbal changes to all: These essays are written for those who like to think, not for the multitude of those who are satisfied with being merely told. To read them understandingly requires neither a mathematician's knowledge of mathematics nor even the rare gift of great mathematical aptitude. What is demanded and all that is demanded is the ability and the willingness to give the matters discussed a measure of disciplined attention.

Some of the essays were evidently suggested by the works of men for whom the author had a high admiration, or with which he disagreed. Thus the essay on mathematics as a culture clue is a critical examination of a well known thesis of Spengler's; *Scientists teach laymen* is in part a critique of works by Hans Reichenbach, H. Levy, and Max Planck; *Mathematics and the science of semantics* is a considered estimate of Korzybski's massive contribution to general semantics in *Science and Sanity;* while *A glance at some of the ideas of Charles Sanders Peirce* incidentally expounds some of Peirce's tardily appreciated innovations in logic and philosophy; *Mathematics and the dance of life* was inspired by Havelock Ellis's book of the same title; and there is an appreciative

account of the work of Pareto, "mathematician, economist, sociologist".

In another essay of great interest, the conclusion is (author's italics) *The Mathematical terms — Relation, Transformation, and Function — are strictly synonomous.* In the essay entitled *Panthetics,* among other things discussed are questions and pseudo-questions, with a rich exhibit of the latter. The essay on *The nature of the doctrinal function and its role in rational thought* explains Keyser's concept of the doctrinal function. It will repay careful study by all interested in postulational systems and the modern abstract approach to several departments of mathematics. One would have thought that this great unifying idea would long since have passed into the common currency of mathematics and epistemology.

California Institute of Technology E. T. Bell

The Strange story of the Quantum. By Banesh Hoffman. Harper and Brothers, New York, 1948. XI 239 pages.

Perhaps the most spectacular and serious single experiment in all of recorded history was the test explosion of the first atomic bomb. The attendant publicity, however, was not sufficiently broad to supply a good picture of the genesis of the fundamental ideas. The general nature of the mathematical and experimental activities that have furnished the devastatingly successful modern physical theories has not been adequately disseminated, and certainly there are some unfortunate misconceptions current with regard to scientific discovery. Science, as we know, is not only the most salient feature of our present civilization, but its potentialities for good and evil are so enormous as to be matters of deep and universal concern. Nevertheless, its recent history has not been awarded sufficient attention, and despite the existence of some excellent books in the popular field, the difficult task of acquainting the general reader with the pertinent facts remains urgent. The general reader has been blessed with many opportunities to learn something of the matters that are relatively easily explained such as elementary mathematics, the lass abstract features of science, the draftman's art, and the "know how" of manufacture and construction; and necessarily these are too often the sole elements of his picture of scientific progress. He has not had an equal opportunity to learn of the role of higher mathematics and quite likely regards higher mathematics as being causally impotent so far as the motivation of scientific discovery is concerned. Certain aspects of radio science, for example, are almost universally known, but how many would ascribe the origin of radio the the mathematical constructions and speculations of James Clerk Maxwell? Fortunately, the present book should prove to be quite effective in restoring the proper balance to the story of scientific progress.

In the reviewers opinion, *The Strange Story of the Quantum* is a superb piece of expository writing. It bears evidence of unusual

linguistic ability combined with a fortunate aptitude for analogy and a good sense of proportion — part of the book is in a light vein while other parts are serious and eloquent; enough of the symbolism and terminology of higher mathematics is present to give a lasting impression of the nature of modern theories without confusing or discouraging the general reader with incomprehensible details. It is made clear that, so far, the secrets of the atom have been wrested from nature by a combination of abstruse mathematics and ingenious physical experiments — that mathematical theory suggests experiment and that the experiments in turn give impetus to the development of the theory. The reader may be surprised to learn that penetration of the unknown is necassarily accompanied by confusion ("It is a poor research worker who insists on fully understanding his own intuition.") and that progress is frequently brought about by accidents. An easy inference from the information presented is that the best way to ensure scientific progress is to encourage research of all kinds. The seemingly impractical may turn out to be devilishly useful.

The general plan of the book is as follows. The prologue presents a description of the famous experiment by Hertz (1887) testing Maxsell's mathematical theory of electricity, magnetism, and light, together with a discussion of various theories of light. Act I, Chapter II is concerned with the birth of the quantum concept and Planck's celebrated radiation formula (1900). Chapter III gives an account of the early contributions of Einstein (1905) to the atomicity of energy. Chapter IV is mainly given over to *interference* and the *photoelectric effect* as related to the wave-particle conflict. Chapter V (The Atom of Niels Bohr) touches on radioactivity, Rutherford's concept of the atom, spectroscopy and the Balmer sequence, the incompatibility of the Rutherford atom and Maxwell's theory, and finally the Nicholson-Bohr formula: $\int p\, dq = nh$. Incidentally, this is the second equation in the book. Chapter VI is concerned with the Zeeman effect, the Stark effect, the Pauli exclusion principle, and the decline of the Bohr theory. Chapter VII is essentially a brief review, while number VIII is a short account of de Broglie's ideas concerning matter waves, and the supporting evidence produced by the Davisson-Germer experiments (electron diffraction patterns). Chapter IX is devoted largely to elementary matrix algebra and Heisenberg's theory. Chapter X introduces the reader very briefly to Dirac's q-numbers and the Poisson brackets and gives a discussion of the relation of Dirac's theory to classical physics and the matrix tabulations of Heisenberg. Incidentally, the first eight chapters will probably be more intelligible to the general reader than numbers IX and X. However, this book is never dull and in these as in preceding chapters the author has been able to capture and pass on much of the excitement of the original discoveries. From chapters XI and XII the general reader will learn of the great importance for modern physics of the mathematical work of Sir William Rowan Hamilton, of how Hamilton's

work paved the way for the successfull Schrödinger wave equation, and finally how Dirac succeeded in displaying the underlying unity in the several apparently diverse theories of the atom. Chapter XIII dwells on some of the conceptual difficulties, such as, Heisenberg's *principle of indeterminacy,* and the meaning of the symbol ψ in Schrödinger's equation. Perhaps the main lesson of Chapter XIV, which is devoted to metaphysical matters related to the new theories, is contained in the following quotations: "We seem to glimpse an eerie shadow world lying beneath our world of space and time; a weird and cryptic world ···"; "··· experiments are a clumsy instrument, afflicted with a fatal indeterminacy which destroys causality. And because our mental images are formed thus clumsily, we may not hope to fashion mental pictures in space and time of what transpires within this deeper world. Abstract mathematics alone may try to paint its likeness." Chapter XV (the epilogue) is taken up for the most part with the contributions of the 1930's — new particles and nuclear fission. "For better or worse, nuclear energy of terrible potency was to be placed in the unready hands of man. ··· . The days of the nightmare are upon us ... ." The closing pages contain some miscellaneous comments clothed in language that is at once beautiful and impelling. "Here in such theories and discoveries is a revelation, all too scant of the mighty wonder that is the universe. Here through the minds of our Einsteins and Bohrs we may dimly sense its structural beauty and cunning intricacy, its soaring poetry and its awe-inspiring grandeur and magnificence, with never a hint of its pain and tragic bestiality. ... . Now is the terrible crisis of our civilization. Now is the fateful hour of high decision. For better or worse, We, the People of the Earth, must choose our future. It can be fine and lovable, gentle and dignified, and filled with joy and wonder and thrilling discovery. Or it can be degraded and obscene, despairing and wretched beyond measure, with death and primitive misery stalking the land unchecked." — This is truly an impressive book.

University of Texas — Homer V. Craig

Six-Place Tables. Seventh Edition. With Explanatory Notes by Edward S. Allen. McGraw-Hill Book Company, Inc., New York, 1947. xxiii + 232 pages. $2.50.

Professor Allen says that the chief improvement in this seventh edition is the expansion of two tables to six places. Those tables are X, Natural Logarithms and XI, Exponential and Hyperbolic Functions.

The book is small enough to be handled easily and to be carried in a coat pocket if one is blessed with pockets. The reviewer likes the ingenious marginal index which enables one to locate the desired table with a minimum of mental and physical effort.

The introduction consists of very clear and careful explanations by Professor Allen, plus two pages of illustrative exercises. On page xix the word "angle" seems to be omitted from the last line, but the meaning

is plain enough. Tables given in the book are those of Squares, Cubes, Square Roots, Cube Roots, Fifth Roots and Powers, Circumferences and Areas of Circles, Common Logarithms of Numbers and of Trigonometric Functions, Natural Trigonometric Functions, Natural Logarithms, Exponential and Hyperbolic Functions, and Integrals.

Wellesley College Marion E. Stark

Fréchet, Maurice. Les Probabilités Associées a Un Système D'Evénements Compatibles Et Dépendants.
Part One: Evénements En Nombre Fini Fixe (viii + 80 pp: 1940)
Part Two: Cas Particuliers Et Applications (131 pp; 1943)
Hermann & Cie, Paris. Paper bound.

These are the first two of three small volumes discussing the probabilities associated with a system of compatible, dependent events. These two volumes give the theory and some applications when there are only a finite number of events. The third, to be published, will extend the theory to an infinite number of events.

Let $A_1, A_2, \cdots, A_m$ be a system of chance events, each of which may or may not occur. Let $p_{a_1 a_2 \ldots a_r}$ be the probability that at least $A_{a_1}, A_{a_2}, \cdots, A_{a_r}$ occur simultaneously. $a_1, a_2, \cdots, a_r$ are r different integers $1 \leq a_i \leq m$. For independent, equally likely events, $p_i = p_j = p$ and $p_{a_1 \ldots a_r} = p^r \cdot p_{a_1 \ldots a_r}$ are the measures of the dependence of the A_i. The fundamental theorem is the following: Let H be any chance function of the events A_i. Then the probability that H occur is a linear function of $p_{a_1 \ldots a_r}$. The moments, distribution function, and other interesting statistics for H can then be computed.

Now we may think of a series of single trials of each A_i as a single trial of the set. Particularly interesting functions H which are considered in detail include R, the "repetition" of A_i, J, the rank of the first A_i to take place, and $H^{(l)}$ the number of runs of length l of consecutive occuring or non-occuring A_i. For example we draw 10 balls from an urn containing white and black balls. A_i is the drawing of a white ball on the ith drawing. R is the number of white balls drawn. J is four if the first three balls are black and the fourth white. $H^{(3)}$ is the number of runs of 3 consecutive balls of the same color. Formulas are given for the statistics such as $P_{[r]}$, the probability that $R = r$, the moments, etc.

Inequalities as well as equalities between the probabilites are discussed, leading to a necessary and sufficient conditions for the existence of a set of events A_i for an arbitrary set of numbers $p_{a_1 \ldots a_r}$ or $p_{[r]}$.

The many examples and applications range from urn drawings to "magic wand" treasure hunting. Coincidence games are discussed in great detail. For example, let m cards be drawn from a bridge deck. A_i occurs if a card of denomination i is the ith card drawn.

Fréchet's style is encyclopedic. The discussions are complete and detailed. Many problems are solved by several methods. The reader arrives at complicated situations by a series of generalizations of simpler situations. Extensive credit is given to other authors for their contributions. References to earlier portions of the work are kept to a minimum by repeating the necessary formulas wherever they are used. The important parts of part one are summarized at the beginning of part two.

Neither the mathematics nor the French is difficult. While a few isolated sections use calculus, in general only algebra is used. There is a detailed table of contents which makes the work quite useful as a reference. Certain sections such as the first part of chapter one, or the discussion of the Bernoullian case in part two will make excellent material for undergraduates wishing to try their hand at mathematical French.

Typographical errors are relatively few and unimportant.

Occidental College Paul B. Johnson

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin, L. J. Adams and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

THE GRAMMAR OF ALGEBRA

E. Justin Hills

The first article in this series entitled "Fundamentals of beginning Algebra" was intended to provide the interested reader with a sufficient knowledge of the language of algebra to enable him to comprehend some of the basic ideas of mathematics and science. It offered a somewhat abbreviated equivalent of Basic English—enough to communicate straightforward, simple concepts. Our algebraic speech was limited to single sentences with subject, verb and object—linear equations in one or two unknowns. In order to be able to express more subtle and more complex ideas, we must, so to speak, learn moods and tenses, conjugations and declensions—in other words, the grammar of algebra.

There is very little in this world that exists as isolated units. Almost everything is dependent upon something else. Our ability to make the United Nations a success depends upon the willingness of participating countries to relinguish a certain degree of national sovereignty. A democratic government presupposes the preparedness and willingness of individuals to assume a just share of responsibility. The maximum speed a car can attain depends upon the horsepower rating of its engine. The area of a circle bears a direct relationship to the length of its radius.

When two quantities are so related that a change in one produces a corresponding change in the other, the latter quantity is said to be a FUNCTION of the former. For instance, the area of a square is a function of the length of the side. The height of a child is a function of his age and increases as he grows older.

These relationships can be expressed in easily translatable shorthand by equations. That is, if a variable y depends on a variable x, these functional relations can be expressed by an equation.

Illustration:

$y = 3x - 8$ implies that the value of y depends on the value assigned to x. That is, if $x = 2$, y will have a related value found by substituting this value of x into the equation. Thus

$$y = 3(2) - 8$$
$$= 6 - 8 = -2$$

There are different kinds of functional relationships. Quantities

may be so related that they either grow together or they diminish together. Such a relationship is said to be a *direct* variation. Note, for example, $p = 4s$ and $C = \pi d$ where p is the perimeter and s the side of a square, and C is the circumference and d the diameter of a circle. If, however, one quantity decreases as the other increases, they are said to vary *inversely*. The intensity of light in any part of a room depends upon the distance from the lamp—the greater the distance, the dimmer the light.

Often there is a multiple relationship. The area A of a rectangle depends on its base b and its altitude a. That is, $A = ba$. If definite values are assigned to two of the letters, the corresponding value of the third letter can be determined. Thus each letter is a function of the other two. In $A = ba$, A is a function of b and a; in $a = \frac{A}{b}$, a is a function of A and b; since, in each case, the value at the left of each equal sign depends on the values assigned to the letters at the right of the equal sign.

Few relationships are as simple as those given above. Most things depend upon not just one but several variables. For example, Newton's law, in physics, states that the force F with which two bodies attract each other varies directly with their masses (M and m) and inversely with the square of the distance d between them. In algebraic shorthand, we state this as follows:

$$F = k\,\frac{Mm}{d^2}$$

where k is a constant determined by the system of measurement used. Again, the amount of interest due on a bank account is a function of the principal invested, the rate of interest and the time it is left in the bank. Thus $I = Prt$, where P = principal invested, r = rate of interest, t = time in years and I = amount of interest due.

Any algebraic equation in more than one unknown is a functional relationship. In an equation, such as $3x = 12$, if each arithmetic number is replaced by a letter, the equation may be written $px = q$, where $p = 3$ and $q = 12$. If we solve for x, namely $x = \frac{q}{p}$, we see that x depends directly on q and inversely on p. That is, $x = \frac{q}{p}$ can be written $x = q \cdot \frac{1}{p}$. Thus, the larger the value of q, the *larger* the value of x if p is unchanged; the larger the value of p, the *smaller* the value of x if q is unchanged. So, if $p = 5$, for instance, and if q takes on the following values, we have:

q	5	10	15	20	25	30
x	1	2	3	4	5	6

Note that as q increases x increases. If $q = 24$, say, and if p takes on the following values, we have:

p	1	2	3	4	5	6
x	24	12	8	6	$4\frac{4}{5}$	4

Thus the value of x varies directly with the value of q, and inversely with the value of p. However the values of q and p can vary at the same time, in which case we say that they vary jointly.

A special group of equations, involving more than one unknown are called *formulas*. These are general statements, in mathematical shorthand, of relationships that exist in geometry, or business, or physics or chemistry. Thus $p = 4s$ and $A = ba$, which were mentioned earlier, are formulas, for each one tells us in algebraic language just what we must do to get the desired quantity: In the former, how to get the perimeter of a square when the side is given; in the latter how to get the area of a rectangle when the sides are given.

Let us consider a few familiar rules which can be expressed in algebraic language:

1) The perimeter of a rectangle is equal to twice the sum of its length and its width.

Solution: Let p, l, and w be defined as perimeter, length, and width respectively. Then

$$p = 2(l + w).$$

If the length of the rectangle is 5 inches, and if its width is 2 inches, the perimeter is 14 inches, since

$$p = 2(5 + 2) = 2 \cdot 7 = 14.$$

2) The amount of a debt is equal to the original loan, called the principal, plus the interest due.

Solution: Let A, P, and I be the amount, the principal, and the interest due respectively. Then

$$A = P + I$$

If the amount is \$384, and the interest earned is \$6, the original principal would be \$378, since

$$384 = P + 6; \text{ so } P = 384 - 6 = 378$$

3) The distance D that a moving object can travel depends jointly on the rate of travel r and the time of travel t. Then

$$D = rt.$$

If the rate is 30 miles per hour (written 30 mi/hr.) and if the time of travel is 8 hours,

$$D = 30 \text{ mi/hr.} \times 8 \text{ hours} = 240 \text{ miles}.$$

4) The rate of travel r of a moving object depends directly on the distance traveled D and inversely on the time of travel t.

$$r = D \cdot \frac{1}{t}, \qquad r = \frac{D}{t}.$$

If the distance traveled is 300 miles and if the time of travel is 10

hours,

$$r = \frac{300 \text{ miles}}{10 \text{ hours}} = 30 \frac{\text{miles}}{\text{hour}}, \text{ or } r = 30 \text{ miles per hour}$$

While the equation gives convenient, compact expression to the idea of dependence, the nature of a dependence is frequently made clearer by use of a graph. For instance an equation tells at a glance whether or not a variation is direct, but the graph shows at a glance whether the quantities increase at the same rate or whether one moves much more rapidly than the other. The graph also indicates whether the relationship is a simple or complex one. For example, a straight line pictures a very simple relationship.

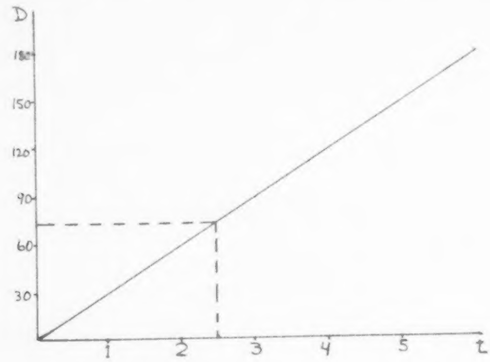


Figure I

The graph shown in Figure I shows how distance increases with time when the rate is fixed at 30 mi/hr. *i.e.*, it is the graph of $D = 30t$, which is a straight line.

Once the graph has been drawn it can be used to find values of either D or t in reference to the other letter. For example: What is the distance traveled in 2½ hours? First locate 2½ on the t axis. Then draw a line parallel to the D axis until it meets the curve. Then draw a line parallel to the t axis until it meets the D axis. This point (namely 75) is the value of D when $t = 2½$. That is: The distance traveled in 2½ hours is 75 miles. In like manner, if $D = 75$, we find graphically that $t = 2½$.

Other groups, such as a circle or parabola, indicate more complex relationships. For each value of one of the variables, there are two corresponding values of the other.

Let us consider the graph of the equation $s = 16t^2$ which gives the relationship between distance and time for a freely falling object. In order to find the value of s, it is necessary to explain the notation t^2. This is mathematical shorthand for the multiplication of two equal or like quantities. It means, simply, t times t, and instead of writing tt or $t \times t$, we write t^2 (read 't squared'). If we wished to indicate

$t \times t \times t$, we should write t^3. The 2 or 3 is called an exponent and is written above and to the right of the base (t, in this case) to indicate the number of times the base is to be used as a factor. To find the value of s in $s = 16t^2$ when $t = 2$, we 'square 2', i.e., multiply 2 by 2 and then multiply the result by 16 thus:

$$16 \times 2^2 \text{ or } 16 \times 4 = 64.$$

(The result of using the same quantity as a factor two or more times is called a *power* of that quantity. For example, 8 is the third power of 2, since $8 = 2 \times 2 \times 2$ or 2^3.) Returning to the equation $s = 16t^2$, having obtained the following pairs of values:

t	0	1	2	3	and so on.
s	0	16	64	144	

In Figure II, we have located each pair of values and then joined

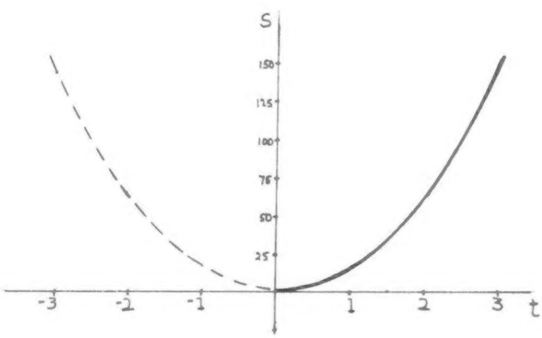


Figure II

them by a smooth curve.

If we consider that time measured from the present backward may be represented by negative values of t, we can find the values of s which correspond to these values of t and complete the graph on the other side of the s axis (shown as a dotted line in the figure). This graph is called a *parabola* and is the characteristic picture of the relationship between two variables in which one appears to the first power only while the other appears to the second poser. (This is known as a *quadratic function*, from the Latin, quadrus = a square.) The graph of the formula for the area of a circle, $A = \pi r^2$, is also a parabola.

An equation in one unknown which contains at least one term of the second power, and no higher power, is called a *quadratic equation*, or an *equation of the second degree*.

Suppose we are faced with the problem of enclosing 8 square feet of area in such a way that the length will be 2 feet more than the width.

Such a problem introduces a quadratic equation, for if the height is x, the base is $x+2$; and since the area is the product of the base and the altitude, we have:

$$x(x+2) = 8 \qquad \text{(altitude} \times \text{base = area)}$$

Or $\quad x \cdot x + 2 \cdot x = 8$

Or $\quad x^2 + 2x = 8.$

Such an equation can be solved by trial, as follows:

Value of x	Value of left side	Value of right side	Does left side equal right side?
1	$1 \cdot 1 + 2 \cdot 1 = 3$	8	No
2	$2 \cdot 2 + 2 \cdot 2 = 8$	8	Yes
3	$3 \cdot 3 + 2 \cdot 3 = 15$	8	No
-1	$(-1)(-1) + 2(-1) = -1$	8	No
-2	$(-2)(-2) + 2(-2) = 0$	8	No
-3	$(-3)(-3) + 2(-3) = 3$	8	No
-4	$(-4)(-4) + 2(-4) = 8$	8	Yes
-5	$(-5)(-5) + 2(-5) = 15$	8	No

Therefore 2 and -4 are the only values of the letter that make both sides have the same number value. Incidently values of the unknown (x here) which satisfy the equation, i.e. make both sides of the equation equal, are called *roots* of the equation.

Such a solution by trial and error is frequently an expensive process from the point of view of time and effort consumed. It is only natural, therefore, that we should seek faster and more efficient methods to solve quadratics.

To do so, we will first consider some *algebraic expressions* which are sums or differences of terms such as $x+3$, $x-5$, x^2-9 and $x^2+8x+15$. The first three expressions are called *binomials* since each contains two terms. The last expression is called a *trinomial* since it contains three terms. If an expression contains four or more terms, such as $x^3 - 8x^2 + 5x - 2$, it is called a *polynomial*.

Expressions, such as $2x$, $x+3$ and $x-5$, have *similar* terms; that is, terms having the same powers of the same letters (here x). In like manner, $2x^2 - 8$, $5x^2$ and x^2+3 have similar terms since each has the same power of the same letter (here x^2). Such expressions can be combined, as we shall now see. When this is done, we have performed *algebraic addition.*

Algebraic addition can best be illustrated by a simple example from arithmetic. If Jack has 3 apples, 2 pears and 5 bananas, and Henry has 2 apples, 1 pear and 3 bananas, together they have 5 apples, 3 pears and 8 bananas. In algebraic notation this might look like this, using a for apples, p for pears and b for bananas:

$$3a + 2p + 5b + 2a + 1p + 3b = 5a + 3p + 8b$$

Just as we combine only like objects in arithmetic, we add only similar terms in algebra.

Illustrations:

1) $$3x - 5y + 8z + 2z + 3y - x = 2x - 2y + 10z$$

2) $$x^2 + 5x - 8 + 2x^2 - 7x + 5 = 3x^2 - 2x - 3$$

Sometimes it is necessary to multiply two binomial expressions together before we can solve a particular quadratic equation. For example, we may have the following problem to solve: What is the side of a square such that when one side is diminished by 3 units and the other increased by 5 units, the area of the resulting rectangle is 48 square units? The algebraic statement of this problem is

$$(x-3)(x+5) = 48.$$

Apparently, before we can even see that this is a quadratic equation we must learn to carry out the implied multiplication of the two parentheses. Let us consider what happens when some arithmetic number is substituted for x in the two parentheses. If, for example, we let $x = 4$, then $(4-3)(4+5) = 1 \times 9 = 9$. Suppose we had multiplied each number in the first parenthesis by each number in the second parenthesis and added the results instead of combining within parentheses first. Then we should have had: $4^2 + 4 \cdot 5 - 3 \cdot 4 - 3 \cdot 5 = 16 + 20 - 12 - 15$ or $36 - 27 = 9$, which is the same result obtained above and indicates the method for multiplying these two binomial expressions, which is as follows:

$$(x-3)(x+5) = x^2 + 5x - 3x - 15 \text{ or } x^2 + 2x - 15.$$

Thus the product of two linear algebraic expressions gives a single quadratic expression. We shall analyze several of these products for by doing so the most important method of solution of quadratic equations will be discovered. Since $3 \times 5 = 15$, 15 is said to have two factors, 3 and 5. Similarly, since $(x-3)(x+5) = x^2 + 2x - 15$, $x^2 + 2x - 15$ is said to have factors, $(x-3)$ and $(x+5)$. We illustrate this idea with a few special products and factors.

Product	*Factors*
$x^2 + 2x$	$x(x+2)$
$x^2 - 9$	$(x+3)(x-3)$
$x^2 + 6x + 9$	$(x+3)(x+3)$
$x^2 + 8x + 15$	$(x+3)(x+5)$
$x^2 - 2x - 15$	$(x+3)(x-5)$

Factoring is thus seen to be a process which reverses multiplication, that is gives the quantities which form the product. By observing each special product, a rule of procedure can be developed for each case. In the first one, we note that x is common in both terms, therefore it can be removed as follows: $x^2 + 2x = x \cdot x + 2 \cdot x = x(x+2)$. In the second

one, we have only two terms, both perfect squares and differing in sign. That being the case, the factors are determined as follows: *One factor is composed of the sum of the square roots and the other of the difference of the square roots*, thus: $x^2 - 9 = (x + 3)(x - 3)$.

If a quadratic expression is set equal to zero, we have a quadratic equation. If in any quadratic equation all terms are put on the same side of the equal sign, it is possible that the expression on that side can be factored.

Illustrations:

1) $x^2 + 2x = 0$ whence $x(x + 2) = 0$

2) $x^2 + 8x + 15 = 0$ whence $(x + 3)(x + 5) = 0$.

A very important principle must now be introduced. Observe that if $2x = 0$, x must equal 0 since 2 cannot equal 0. But if $xy = 0$, either $x = 0$, or $y = 0$ or both x and y equal zero will suffice. That is, if the product of two or more factors is zero, at least one factor must equal zero. Thus, our problem resolves itself into finding the values of x that make the various factors zero. That is, if $(x + 3)(x + 5) = 0$, either $x + 3 = 0$, or $x + 5 = 0$, or both equal zero. In order that the factor $(x + 3)$ be equal to zero, x must have a value of -3. Similarly, $(x + 5)$ will be zero if $x = -5$. Thus, both the values $x = -3$ and $x = -5$ satisfy the equation, $x^2 + 8x + 15 = 0$, which is gotten by multiplying $(x + 3)$ by $(x + 5)$. We now check this directly by substituting these values for x in this equation.

If $x = -3$, we have

$$(-3)^2 + 8(-3) + 15 = 0$$
$$9 - 24 + 15 = 0$$

If $x = -5$, we have

$$(-5)^2 + 8(-5) + 15 = 0$$
$$25 - 40 + 15 = 0$$

Factoring is thus seen to be an important approach to the solution of quadratic equations.

Further illustrations:

(1) What values of x satisfy the equation: $x^2 = 9$?

Solution: Rewrite the equation in the form $x^2 - 9 = 0$. Factor the left side: $(x + 3)(x - 3) = 0$. Set each factor equal to zero and solve for x: $x + 3 = 0$, so $x = -3$; $x - 3 = 0$, so $x = 3$. Checking $(-3)^2 = 9$ and $(3)^2 = 9$.

2) What values of x satisfy the equation: $x^2 + 6x + 9 = 0$?

Solution: Factor the left side: $(x + 3)(x + 3) = 0$. Since both factors are the same, set one of them equal to zero and solve for x: $x + 3 = 0$, $x = -3$. Since $x = -3$, and only -3, there is only one value of x that satisfies the given equation, but this value is thought of as being a repeated root.

This factoring method of solution of quadratic equations is certainly faster and more interesting than was the trial method. Unfortunately it does not always work. There are many quadratic expressions that cannot be factored just as there are many arithmetic numbers that cannot be factored, such as 7, 11, 17, and so on. Let us consider a method which can be used when the equation is not factorable.

We return to examine some of the equations already solved, in order to devise a method which will help us to solve quadratics which are not factorable. If $x^2 = 9$, take the square root of both sides. Then $x = \pm 3$, where $\pm$ means $+$ or $-$. If $x^2 + 6x + 9 = 0$, we can put it into the form $(x + 3)^2 = 0$, and take the square root of both sides. Then $x + 3 = \pm 0$, or merely 0. Transposing, we have $x = -3, -3$.

This approach can be used on the quadratic which is not factorable. Evidently, what is required is that the equation be balanced in such a way that the left member is made a perfect square. Take, for example, the equation $x^2 + 6x + 5 = 0$. The product $x^2 + 6x$ is composed of the factors x and $x + 6$. These may be considered to be sides of a rectangle which we wish to convert into a square:

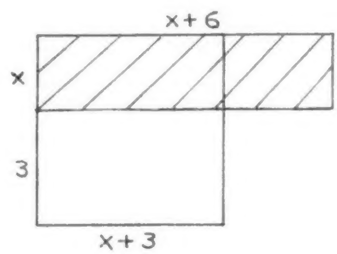


Figure III

By taking 3 units from the long side and adding 3 units to the shorter side, we obtain a square of side $x + 3$. The area of this square is $(x + 3)^2$ or $(x + 3)(x + 3)$ which is $x^2 + 6x + 9$. Comparing this with the left member of the original equation, we see that by subtracting 5 from both members of the equation then adding 9 to $x^2 + 6x$, the left side can be made a perfect square. However, since the balance of the equation must not be destroyed, when we add 9 to the left member we must also add 9 to the right member, thus:

$$x^2 + 6x + 9 = -5 + 9$$

Or, $$x^2 + 6x + 9 = 4$$

Then taking the square root of both sides, as before:

$$(x + 3) = \pm 2$$
$$x = \pm 2 - 3$$

and $x = -1$ or -5.

This method enables us to arrive at a formula for solving any quadratic equation, whether or not it is factorable. Let us consider the characteristics of a trinomial like $x^2 + 6x + 9$ which is a perfect square. Note that both first and last terms are squares (x^2 and 9) and that the middle term is twice the product of their square roots ($2 \cdot 3 \cdot x$ or $6x$). Returning to our rectangle, we see that, in order to build a perfect square, all we have to do is split the excess of one side over the other equally between the two sides, as in Figure IV, forming the square as indicated by the dotted lines. Then we have

$$(x + ½b)(x + ½b) = x^2 + bx + ¼b^2$$

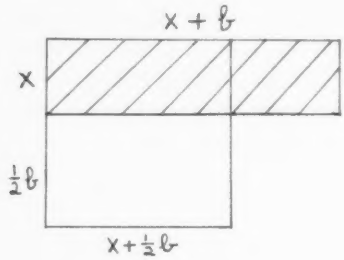


Figure IV

In other words, to make $x^2 + bx$ a perfect square, we take ½ of b (the coefficient of x) square it, and add it to $x^2 + bx$. Now we are prepared to solve any quadratic equation. First let us apply this method to one more illustrative example:

$$x^2 + 8x + 7 = 0$$

$x^2 + 8x$ can be made a perfect square by adding the square of half the coefficient of x, *i.e.*, 4^2 or 16. Rewriting the original equation, transposing the 7, (*i.e.*, subtracting 7 from both sides), we obtain: $x^2 + 8x = -7$. Adding 16 to both sides, we obtain: $x^2 + 8x + 16 = -7 + 16$, or $(x + 4)^2 = 9$. Taking the square root of both sides: $x + 4 = \pm 3$ whence $x = \pm 3 - 4$ or $x = -1$, or -7.

In an equation, such as $x^2 + 8x + 15 = 0$, if the arithmetic numbers are replaced by letters, we can have $x^2 + bx + c = 0$. Then,

Putting c on the right side $\qquad x^2 + bx = -c$

Adding to both sides the square of one-half the coefficient of x $\qquad x^2 + bx + \frac{b^2}{4} = -c + \frac{b^2}{4}$

Simplifying both sides $\qquad (x + \frac{b}{2})^2 = \frac{b^2 - 4c}{4}$

Taking the square root of both sides $\qquad x + \frac{b}{2} = \pm \frac{\sqrt{b^2 - 4c}}{2}$

Transposing $\frac{b}{2}$ to the right side $\qquad x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$

Combining terms on the right side $\qquad x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

The right side of the equal sign is an algebraic expression involving the coefficient of x, b and the constant, c. It can be used to solve any quadratic equation.

Illustrations:

1) $x^2 + 8x + 15 = 0$ What is the value of x?
Solution: $b = 8$ and $c = 15$. Therefore:

$$x = \frac{-8 + \sqrt{64 - 60}}{2} = \frac{-8 + 2}{2} = -3, \quad \text{and}$$

$$x = \frac{-8 - \sqrt{64 - 60}}{2} = \frac{-8 - 2}{2} = -5.$$

2) $x^2 - 9 = 0$ What is the value of x?
Solution: $b = 0$ and $c = -9$. Therefore:

$$x = \pm \frac{\sqrt{36}}{2} = \pm \frac{6}{2} = \pm 3.$$

3) $x^2 + 5x - 3 = 0$ What is the value of x?
Solution: $b = 5$ and $c = -3$. Therefore:

$$x = \frac{-5 \pm \sqrt{25 + 12}}{2} = \frac{-5 \pm \sqrt{37}}{2}$$

Here we have solved a quadratic equation which could not be factored. Note that the solution results in an indicated root which cannot be found exactly ($\sqrt{37}$). Since $\sqrt{37}$ is approximately 6, ($\sqrt{36} = 6$), approximate values of x are

$$x = \frac{-5 + 6}{2} = \frac{1}{2} \quad \text{and} \quad x = \frac{-5 - 6}{2} = -5\tfrac{1}{2}.$$

4) $x^2 + 5x + 8 = 0$ What is the value of x?
Solution: $b = 5$ and $c = 8$. Therefore:

$$x = \frac{-5 \pm \sqrt{25 - 32}}{2} = \frac{-5 \pm \sqrt{-7}}{2}$$

This time we have a minus sign under the radical sign. Thus the solution of quadratic equations introduces two types of numbers we have not dealt with before.

The first numbers, a child meets, are the whole numbers he uses for

counting objects. As his knowledge increases and he learns to share things, he comes to use fractions. Now we have arrived at a point where these two types of numbers no longer suffice. We have numbers such as $\sqrt{3}$ and $\sqrt{37}$, which cannot be expressed exactly as the quotient of two integers — these we call *irrational* numbers, and we also have numbers such as $\sqrt{-7}$ which are called, unfortunately, *imaginary* numbers.

The expression $\sqrt{37}$ is a radical of the *second order* since $\sqrt{37}$ means the square (second) root of 37. It could be written $\sqrt[2]{37}$, but, since most radicals are of the second order, the 2 is omitted for convenience. The sign $\sqrt{\ }$ is a contraction of r and the vinculum $\overline{\ \ }$.

Radicals of the second order are frequently met in geometric solutions. For example, the side of a square whose area is known is given by the formula: $s = \sqrt{A}$. Thus, if $A = 4$, $s = \sqrt{4} = 2$, since $2 \times 2 = 4$; but if $A = 5$, $x = \sqrt{5}$. Since there is no arithmetic number which when multiplied by itself gives 5, we will need to find an approximate value for the square root of 5.

There are several methods of approximating square roots. One of the most interesting and most practicable is the mechanics' rule. Suppose, for example, that we wish to find $\sqrt{175}$. We can consider that 175 is the area of a square and that we are required to find the approximate length of the side of such a square. Since $13^2 = 169$ and $14^2 = 196$, the side is evidently between 13 and 14. Suppose we estimate it to be 13. If our estimate were exactly right, then the result of dividing 13 into 175 should be 13. Since, however, we have underestimated the length of the side, the result will be larger than 13; on the other hand, if we had overestimated it, the quotient would be smaller than the desired side. Dividing 175 by 13 we obtain 13.4615. By averaging 13 and 13.4615, we obtain a much closer estimate of the length of the side, namely $\frac{13 + 13.4615}{2}$ or 13.231. To sharpen our approximation, we repeat the process using 13.231 as our divisor, thus: 175 divided by 13.231 gives 13.2265. Again averaging divisor and quotient: $\frac{13.231 + 13.2265}{2} = 13.2287$. This result is accurate to four significant figures. If greater accuracy is desired, the process may be repeated.

Many radicals can be simplified so that the radicand becomes a small number such as 2 or 3 or 5. Then, by learning the approximate values of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$, we can quickly arrive at desired results. For example, since $8 = 4 \times 2$,

$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$$

In like manner, $12 = 2\sqrt{3}$, since $12 = 4 \times 3$.

In an expression such as $\frac{6 + 4\sqrt{3}}{2}$, the 2 can be cancelled into both the 6 and 4 giving $3 + 2\sqrt{3}$ as a result. That is

$$\frac{6 + 4\sqrt{3}}{2} = \frac{2(3 + 2\sqrt{3})}{2} = 3 + 2\sqrt{3}$$

Similar radicals, such as $2\sqrt{3}$ and $5\sqrt{3}$, can be added or subtracted. That is, $2\sqrt{3} + 5\sqrt{3} = (2 + 5)\sqrt{3} = 7\sqrt{3}$ just as we add $2x + 5x = 7x$.

Radicals of the same order can be multiplied or divided. For example: $\sqrt{9}\times\sqrt{4} = \sqrt{9\times 4} = \sqrt{36} = 6$, or, taking the square roots separately as a check $\sqrt{9}\times\sqrt{4} = 3\times 2 = 6$.

Illustrations:

1) $$\sqrt{8}\times\sqrt{2} = \sqrt{8\times 2} = \sqrt{16} = 4$$

2) $$\sqrt{12}\div\sqrt{3} = \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2.$$

We mentioned earlier the use of the exponent to indicate a power of a number. (See the introductory discussion about quadratics.) It was noted then that a^2 means $a\cdot a$ and that $a^3 = a\cdot a\cdot a$, etc. The rules for carrying out multiplication and division of quantities which involve exponents are quite simply deduced from an examination of the notation, thus, $a^2\times a^3 = a\cdot a\times a\cdot a\cdot a$ or a^5 which means we merely add the exponents. This seems reasonable when we consider that $a^2\times a^3$ means that a is to be used as a factor twice and then three times more or a total of five times. In division, three types of results are met.

Illustrations:

1) $$a^5\div a^2 = \frac{a^5}{a^2} = \frac{aaa\cancel{a}\cancel{a}}{\cancel{a}\cancel{a}} = a^3, \text{ or } a^{5-2} = a^3$$

2) $$a^2\div a^2 = \frac{a^2}{a^2} = \frac{aa}{aa} = 1$$

3) $$a^2\div a^5 = \frac{a^2}{a^5} = \frac{aa}{aaaaa} = \frac{1}{aaa} = \frac{1}{a^3}.$$

Through 2), in the last illustrations, *zero* exponents are introduced into our mathematical vocabulary. Since $a^2\div a^2 = 1$ and $a^2\div a^2 = a^0$, it is convenient to define the 0 power of any quantity to be equal to 1. Similarly, through 3), negative exponents are introduced. Since by 3) $a^2\div a^5 = \frac{1}{a^3}$ and $a^2\div a^5 = a^{2-5} = a^{-3}$, we define the negative exponent to be the reciprocal of the same power with a positive exponent, i.e., $a^{-n} = \frac{1}{a^n}$, where n can be any number.

Since our work with positive integral exponents has forced us to note and define the zero and negative exponents, the question arises: Can we also have fractional exponents, and, if so, what possible meaning can be attached to them? Does $a^{1/2}$ mean anything? If it does perhaps we can arrive at that meaning by applying the rules we have just developed for integral exponents. Let us multiply $a^{1/2}$ by $a^{1/2}$. Adding the exponents, just as we did integral exponents, we have $a^{1/2}\cdot a^{1/2} = a$. This would seem to indicate that $a^{1/2}$ is one of the two equal factors of a. But that is precisely our definition of the square root of a. Since it would be illogical to have two different quantities with the same meaning, we define $a^{1/2}$ to be the same as $\sqrt{a}$. In like manner, since $a^{1/3}\cdot a^{1/3}\cdot a^{1/3} = a$, we shall say that $a^{1/3}$ is the cube root of a (i.e. $\sqrt[3]{a} = a^{1/3}$).

This definition of fractional exponents and the application of the same rules for multiplication and division as were used for integral

exponents enable us to devise a method for multiplying and dividing radicals of different order. Note that, in order to add or subtract fractional exponents, they must be changed to a common denominator.

Illustrations:

1) $\sqrt{2} \times \sqrt[3]{2} = 2^{1/2} \times 2^{1/3} = 2^{1/2 + 1/3} = 2^{5/6} = \sqrt[6]{2^5} = \sqrt[6]{32}$

2) $\sqrt{3} \times \sqrt[3]{2} = 3^{1/2} \times 2^{1/3} = 3^{3/6} \times 2^{2/6} = \sqrt[6]{3^3 \times 2^2} = \sqrt[6]{108}$

3) $\sqrt[3]{4} \div \sqrt{2} = 4^{1/3} \div 2^{1/2} = 4^{2/6} \div 2^{3/6} = \sqrt[6]{\frac{16}{8}} = \sqrt[6]{2}$

The laws for the multiplication and division of quantities which involve exponents give us a powerful tool for simplifying computation. Since we add exponents when we multiply, it would be possible to replace the process of multiplication by addition if we could express the numbers to be multiplied by means of a common base with suitable exponents. For example, we can multiply 16 by 64 by changing 16 to read 2^4 and 64 to read 2^6. Then $2^4 \times 2^6 = 2^{10}$. We see now that it is necessary to have a table of the powers of 2 so that we can look up 2^{10} and find the answer to our problem quickly. Such a table is called a table of logarithms. *Logarithms are exponents.*

Since our number system is a decimal system, *i.e.*, expressed in terms of the powers of 10, logarithms most commonly used are simply the exponents which indicate the powers of 10 necessary to represent all of the numbers. Thus, it is found in a log table that $10^{1.1761} = 15$ and $10^{2.2625} = 183$ and if we wish to multiply 15×183, we add 1.1761 and 2.2625 and look this result up in the table of logarithms to find what number is represented by $10^{3.4386}$. Similarly, if we wish to divide 183 by 15, we subtract their respective logarithms and look up the result in the table. It is interesting to note, in passing, that logarithms can be used to find the square root of a number by dividing the logarithm of the number by 2 and referring to the table. In the example worked by mechanics' rule, we found $\sqrt{175}$. This could be done by logarithms as follows:

From the table, $10^{2.5596} = 175$

Therefore, $\log 175 = 2.5596$

Dividing the log by 2, we get 1.2798.

Looking this up in a table, we find that $10^{1.2791} = 13.22$ and that $10^{1.2799} = 13.23$. Therefore the square root of 175 is very nearly 13.23.

Thus we see that the fundamental principle of the epoch-making invention of logarithms is very simple. Since it simplifies computations that must be made in finance, insurance, mathematics and other physical sciences, the ability to use logarithms skillfully is a highly desirable acquisition.

We have used the problem of area as applied to squares and rectangles as a basis for our discussion of quadratics. A consideration of the formulas for volume ($V = s^3$, $V = \frac{4}{3}\pi r^3$) shows that we are also concerned with equations of the third degree.

Let us examine equations of the third degree, such as

$$x^3 - 5x^2 + 2x + 8 = 0.$$

Such an equation can be solved (*i.e.*, the values of the unknown that satisfy the equation can be found) by trial as follows:

Value of x	Value of left side	Value of right side	Does left side equal right side
1	$1 - 5 + 2 + 8 = 6$	0	No
-1	$-1 - 5 - 2 + 8 = 0$	0	Yes
2	$8 - 20 + 4 + 8 = 0$	0	Yes
-2	$-8 - 20 - 4 + 8 = -24$	0	No
4	$64 - 80 + 8 + 8 = 0$	0	Yes
-4	$-64 - 80 - 8 + 8 = -144$	0	No
8	$512 - 320 + 16 + 8 = 216$	0	No
-8	$-512 - 320 - 16 + 8 = -840$	0	No

Therefore -1, 2 and 4 are the only values of x that make both sides have the same number value. As a matter of fact, one can show that such an equation has only as many roots, *i.e.*, values of x which make the sides identical, as the order of the equation. That is, a third order equation (called a cubic equation) has three roots.

Again it will be to our advantage to determine methods of solution that are shorter than the trial method. We start with the trial method, but just as soon as an exact value of the unknown has been found, if such is possible, we factor the given equation. If no exact value is found by trial, it is necessary to use special methods as will be shown later. Consider the equation: $x^3 - 5x^2 + 2x + 8 = 0$. If $x = -1$, the left side of the equal sign equals the right side. $x + 1$ is therefore a factor.

We must divide $x^3 - 5x^2 + 2x + 8$ by $x + 1$ to find the other factors, just as we would divide 1655 by 5, once we note that 5 is a factor of 1655. This is done as follows:

$$\begin{array}{r}
331 \\
5\,/\overline{1655} \\
\underline{15} \\
15 \\
\underline{15} \\
5 \\
\underline{5}
\end{array}
\qquad\qquad
\begin{array}{r}
x^2 - 6x + 8 \\
x + 1\,/\overline{x^3 - 5x^2 + 2x + 8} \\
\underline{x^3 + x^2} \\
- 6x^2 + 2x \\
\underline{- 6x^2 - 6x} \\
8x + 8 \\
\underline{8x + 8}
\end{array}$$

Steps of solution: Since $x^2 \cdot x = x^3$, put x^2 in the quotient. Then multiply $x + 1$ by x^2 putting the product below the dividend, here $x^3 - 5x^2 + 2x + 8$. Subtract, being careful of signs. Repeat this process with the reduced expressions until there is no expression left.

Thus $x^3 - 5x^2 + 2x + 8 = (x + 1)(x^2 - 6x + 8) = 0$.

Now by factoring, we find that $x^2 - 6x + 8 = (x - 4)(x - 2)$. Therefore $x^3 - 5x^2 + 2x + 8 = (x + 1)(x - 4)(x - 2) = 0$. If each factor is set equal to zero, we find that $x = -1$, $x = 4$ and $x = 2$. Thus it was not necessary to use the trial method to find *all* the values of x that satisfy the equation. (If the quadratic factor cannot be factored, the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$, may be applied to find a pair of irrational roots or a pair of imaginary roots.)

In the case of a cubic equation, one root must be real even though the other roots are imaginary. So at least one real root of a cubic equation can be found approximately by using graphical methods.

To show how graphical methods can be used to determine the real roots of an equation, let us first consider an example $x^3 - 5x^2 + 2x + 8 = 0$. Consider the equation $y = x^3 - 5x^2 + 2x + 8$. Now set up the following pairs of values:

x	-3	-2	-1	0	1	2	3	4	5	$6 \cdots$
y	-70	-24	0	8	6	0	-4	0	18	$56 \cdots$

We then locate each pair of values on a diagram, as in Figure V, and join them by a smooth curve.

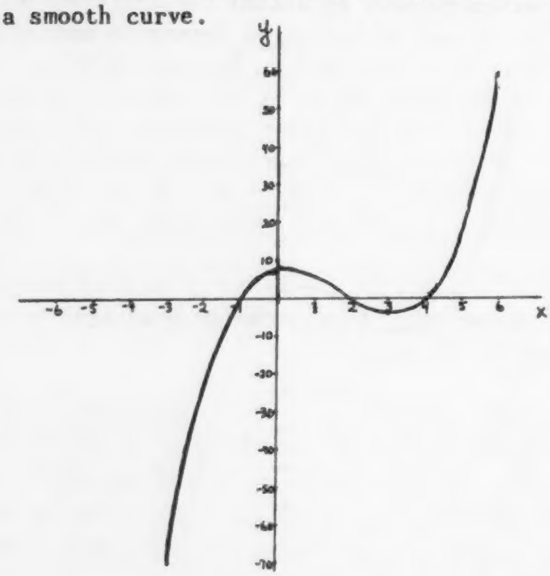


Figure V

Observe that where the graph crosses the x axis (at $x = -1$, $x = 2$ and $x = 4$), y has a value of zero. But when $y = 0$, we have the original cubic equation. Hence, the roots of a cubic equation in x are the values of x which correspond to $y = 0$ or the points of intersection with the x axis.

Any irrational root of a cubic equation in x can be approximated

by estimating where the graph representing such an equation crosses the x axis.

The graph suggests something useful for solving equations that cannot be factored, for when it crosses the x axis the values of y just before and just after intersection must differ in sign. Let us consider the equation $x^3 + x - 3 = 0$. Let $y = x^3 + x - 3$. The following pairs of values can be set up:

x	-2	-1	0	1	2	3	$\cdots$
y	-13	-5	-3	-1	7	27	$\cdots$

Locate each pair of values on a diagram, as in Figure VI, and join them by a smooth curve.

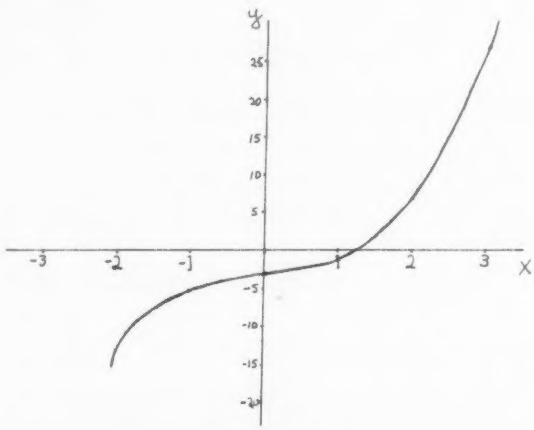


Figure VI

Note that the curve crosses the x axis at only one point. This indicates that the other two roots must be imaginary.

Let us approximate the irrational root shown in Figure VI. When $x = 1$, $y = -1$, and when $x = 2$, $y = 7$. The real root must lie between $x = 1$ and $x = 2$. Furthermore it lies nearer $x = 1$ than it does to $x = 2$ since the y value of the former is closer to 0. If we let $x = 1.2$, $y = -0.072$, and if $x = 1.3$, $y = 1.487$. Thus the real root is slightly greater than 1.2. If greater accuracy is desired, we let x take on values such as $x = 1.21$ and so on and compare corresponding values of y. Thus we can obtain a root to any desired degree of accuracy.

Algebra, it will be noted, is concerned primarily with the solution of equations. All of the processes and devices we have discussed have been directed, ultimately, toward that end. This is quite natural, since the equation is the scientist's means of expressing laws which govern the behavior of matter from the growth of bacteria to the movements of the planets. For this purpose, the solution of equations is pursued in some form or other in many branches of higher mathematics. We hope this is an understandable introduction to such further studies.

PROBLEMS AND QUESTIONS

Edited by

C. G. Jaeger, H. J. Hamilton and Elmer Tolsted

This department will submit to its readers, for solution, problems which seem to be new, and subject-matter questions of all sorts for readers to answer or discuss, questions that may arise in study, research or in extra-academic applications.

Contributions will be published with or without the proposer's signature, according to the author's instructions.

Although no solutions or answers will normally be published with the offerings, they should be sent to the editors when known.

Send all proposals for this department to the Department of Mathematics, Pomona College, Claremont, California. Contributions must be typed and figures drawn in india ink.

SOLUTIONS

No. 18. Proposed by Julius Sumner, Dillard University.

A smooth circular hoop rests on a smooth horizontal table. A small marble is to be projected from a point A on the inner side of this hoop so as to return to point A after two reflections, or rebounds. If e is the coefficient of restitution, and both friction and rolling are neglected, find the angle between the first path and the radius drawn to A.

Solution by Howard Eves, San Francisco, California.

Let the hoop be represented in Fig. 1 (see next page) by the circle with center O, and the path of the marble by the sides AB, BC and CA of the triangle ABC circumscribed by the circle. The radii OA, OB and OC subtend pairs of equal angles, α, β and γ, at the sides of the triangle ABC, which are chords of the circle, and the sum $\alpha+\beta+\gamma$ equals one right angle.

Since the hoop is assumed not to be perfectly elastic, the angle of incidence at the point of impact will not be equal to the angle of reflection. In Fig. 2, let the marble strike the inside of the hoop with a velocity v, and let its path be inclined at an angle θ to the radius R of the circle. The velocity v is decomposed into a radial component $v\cdot\cos\theta$ and a tangential component $v\cdot\sin\theta$. Since no friction or rolling is assumed to occur, the tangential velocity component will remain unchanged after the rebound, while the radial component, besides being reversed, must be multiplied by the coefficient of restitution $e<1$, becoming $e\cdot v\cdot\cos\theta$.

Let the rebound velocity equal v', and the angle of reflection equal θ'. Then the components of the rebound velocity will be $v'\cdot\cos\theta'$ and $v'\cdot\sin\theta'$. It follows that

$$(1)\quad v'\cdot\sin\theta' = v\cdot\sin\theta \quad\text{and}\quad (2)\quad v'\cdot\cos\theta' = e\cdot v\cdot\cos\theta.$$

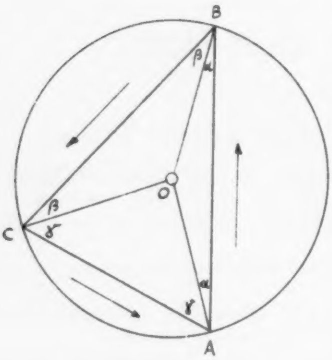


Fig. 1

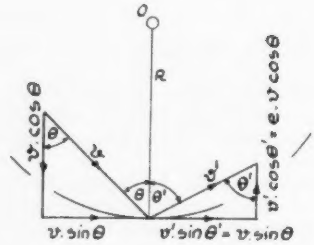


Fig. 2

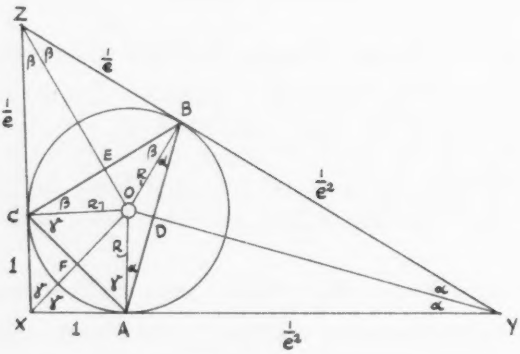


Fig. 3

Dividing (1) by (2) we obtain

$$(3) \qquad \tan\theta' = \frac{1}{e}\tan\theta,$$

which gives the relationship between the angles of incidence and reflection.

Applying this result to the triangular path of the marble in Fig. 1, we obtain:

$$(4) \qquad \tan\beta = \frac{1}{e}\tan\alpha; \quad \tan\gamma = \frac{1}{e}\tan\beta; \quad \therefore \tan\gamma = \frac{1}{e^2}\tan\alpha.$$

The problem now reduces to finding three angles whose sum is 90°, and whose tangents are in the ratio $1 : \frac{1}{e} : \frac{1}{e^2}$.

In Fig. 3, let the hoop again be represented by the circle (O), the path of the marble by the triangle ABC, and the angles which its sides make with the radii OA, OB and OC by α, β, and γ, whose tangents are assumed to obey the relationships of equations (4).

At A, B and C, draw the tangents to the circle (O), which intersect at X, Y and Z. Also draw OX, OY and OZ, which intersect AB, BC and CA at D, E and F, respectively.

The right-angled triangles OAD and OAY are similar, having $\angle AOD$ in common. Therefore $\angle OYA = \angle OAD = \alpha$. Similarly, $\angle OZC = \beta$, and $\angle OXA = \gamma$. Now $\tan\alpha = R/AY$, and $\tan\gamma = R/AX$; therefore,

(5) $\tan\gamma/\tan\alpha = AY/AX$,

and, if AX is made equal to unity, $\tan\gamma/\tan\alpha = AY$. But by (4). $\tan\gamma/\tan\alpha = 1/e^2$.

Therefore,

(6) $AY = 1/e^2$.

Similarly, it can be shown that

(7) $CZ = 1/e$.

Since both tangents from a point to a circle are equal, the semi-perimeter of the triangle XYZ is given by

(8) $s = 1 + 1/e + 1/e^2$.

Hence

$$\tan \tfrac{1}{2} < YX = \tan\alpha = \sqrt{\frac{(S-YZ)(S-XY)}{S(S-XZ)}}$$

$$= \sqrt{\frac{(1+\frac{1}{e}+\frac{1}{e^2}-\frac{1}{e}-\frac{1}{e^2})(1+\frac{1}{e}+\frac{1}{e^2}-1-\frac{1}{e^2})}{(1+\frac{1}{e}+\frac{1}{e^2})(1+\frac{1}{e}+\frac{1}{e^2}-1-\frac{1}{e})}} = \sqrt{\frac{\frac{1}{e}}{\frac{1}{e^2}(1+\frac{1}{e}+\frac{1}{e^2})}} = \sqrt{\frac{e^2}{1+e+\frac{1}{e}}};$$

$$\text{so } \tan\alpha = e\sqrt{\frac{1}{1+e+\frac{1}{e}}}, \text{ or } \alpha = \tan^{-1} e\sqrt{\frac{1}{1+e+\frac{1}{e}}}$$

Solved also by C. W. Trigg, Los Angeles, and Howard Eves, Corvallis, Oregon.

No. 19. Proposed by V. Thébault, Tennie, Sarthe, France.

Find a perfect square such that the numbers formed by its digits, taken in sets of three marked off from the right, and by its square root, form three consecutive terms of an arithmetic progression whose common difference is r^2. i.e. if $n^2 = \overline{abcdef}$, then $\overline{abc}$, $\overline{def}$, n form an arithmetic progression.

Solution by C. W. Trigg, Los Angeles City College.

$n^2 = \overline{abcdef} = r^3P + Q = r^3(n - 2r^2) + (n - r^2)$. It follows that $n^2 - n(r^3+1) + r^2(2r^3+1) = 0$, so $n = [r^3 + 1 \pm \sqrt{(r^3+1)^2 - 4r^2(2r^3+1)}]/2$. In order that the quantity under the radical may be a perfect square, $r > 7$ and $r \neq 5k+1$. The only value of $r < 50$ meeting these conditions is $r = 10$. This gives $n = 725$ or 276, and the two solutions: $525625 = (725)^2$ and, if zero be an admissible initial digit, $076176 = (276)^2$.

By dropping the restriction $d = r^2$, requiring that $a \neq 0$, and taking all possible orders of the pnogression in the scale of 10, we have $n^2 = (10)^3P + Q$, so $P < n$ and $317 \leq n \leq 999$. Hence there are three possible orders:

1). P, Q, n. Then $2Q = P + n$. Eliminating Q, we have $n(2n-1) = 2001\,P = (3)(23)(29)P$. Within the established limits there is only one solution of this equation, namely $n = 725$, $P = 525$. Therefore the unique solution is $525625 = (725)^2$ for which d happens to be $100 = (10)^2 = r^2$.

2). P, n, Q. Then $2n = P + Q$. Eliminating Q, then $n(n-2) = 3^3(27)P$. There are but two solutions of this equation: $n = 594$, $P = 352$ to which corresponds $352836 = (594)^2$; and $n = 407$, $P = 165$, so $165649 = (407)^2$. In both cases, $d = 242 = 2(11)^2 = 2(r+1)^2$.

3). Q, P, n. Then $2P = n + Q$. Eliminating Q, $n(n+1) = 2(3)(167)P$. There are five solutions of this equation, one of which gives a negative Q. The other four give $250000 = (500)^2$, $251001 = (501)^2$, $446224 = (668)^2$ and $695556 = (834)^2$.

No. 20. Proposed by V. Thébault, Tennie, Sarthe, France.

In any tetrahedron $ABCD$, of centroid G, for which the tetrahedron $GABC$ is trirectangular at G, show that the relations

$$m_a^2 + m_b^2 + m_c^2 = 11\,m_g^2$$

$$(ABD)^2 + (BCD)^2 + (CAD)^2 = 11(ABC)^2$$

hold between the lengths m_a, m_b, m_c, m_g of the medians of the tetrahedron $GABC$ drawn from A, B, C, G and the areas $(ABD), \cdots$, of the faces $ABD, \cdots$, of the tetrahedron $ABCD$.

Solution by Howard Eves, Oregon State College.

We shall use the known fact that the square of the median issued from a given vertex of a tetrahedron is equal to the arithmetic mean of the squares of the three edges issued from the same vertex diminished by one ninth the sum of the squares of the remaining edges (see Altshiller-Court, *Modern Pure Solid Geometry*, art. 187, p. 57). Applying this theorem to the medians m_a, m_b, m_c of the tetrahedron $GABC$ and keeping in mind that G–ABC is trirectangular we have

$$\begin{aligned} m_a^2 + m_b^2 + m_c^2 &= (5AB^2 + 5BC^2 + 5CA^2 - AG^2 - BG^2 - CG^2)/9 \\ &= 11(AG^2 + BG^2 + CG^2)/9 = 11(3AG^2 + 3BG^2 + 3CG^2 - AB^2 - BC^2 - CA^2)/9 \\ &= 11\,m_g^2. \end{aligned}$$

Now, designating AB, BC, CA, DA, DB, DC by a, b, c, a', b', c' respectively, and using the relations (*ibid.*, ex. 3, p. 57)

$$2(a^2+b^2) = c^2 + c'^2,\quad 2(b^2+c^2) = a^2 + a'^2,\quad 2(c^2+a^2) = b^2 + b'^2,$$

we have, by Heron's formula,

$$\begin{aligned} &(ABD)^2 + (BCD)^2 + (CAD)^2 \\ &= 2c^2a'^2 + 2c^2b'^2 + 2a'^2b'^2 - c^4 - a'^4 - b'^4 \end{aligned}$$

$$2a^2b'^2 + 2a^2c'^2 + 2b'^2c'^2 - a^4 - b'^4 - c'^4$$

$$+ 2b^2c'^2 + 2b^2a'^2 + 2c'^2a'^2 - b^4 - c'^4 - a'^4$$

$$= 11(2a^2c^2 + 2c^2b^2 + 2b^2a^2 - a^4 - b^4 - c^4)$$

$$= 11(ABC)^2.$$

Solved also by C. W. Trigg, Los Angeles, California.

No. 22. Proposed by Pedro A. Piza, San Juan, Puerto Rico.

Let x and n be any two positive integers and let $\Sigma^n x^2$ stand for the nth iterated summation of all the squares from 1 to x^2 inclusive. For instance $\Sigma\, 4^2 = 30$, $\Sigma^2\, 4^2 = 50$, (that is, the sum of the sums of all the squares from 1 to 16 inclusive), and $\Sigma^5\, 4^2 = 156$ (that is, the sum of the sums of the sums of the sums of the sums of all the squares from 1 to 16 inclusive). Prove that in general

$$\Sigma^n\, x^2 = x(x+1)(x+2)\cdots(x+n)(2x+n)/(n+2)!$$

Solution by Howard Eves, Oregon State College.

Set

$$f(x,n) = x(x+1)(x+2)\cdots(x+n)(2x+n)/(n+2)!$$

We must show that

$$f(x,1) = \sum_{y=1}^{x} y^2, \qquad f(x,n) = \sum_{y=1}^{x} f(y,n-1),\ n > 1.$$

Now it is easily shown that

(1) $f(1,n) = 1,$

(2) $f(x,1) = x(x+1)(2x+1)/6 = 1^2 + 2^2 + \cdots + x^2,$

(3) $f(x,n) - f(x-1,n) = f(x,n-1).$

(4) $\sum_{y=1}^{x} f(y,n-1) - \sum_{y=1}^{x-1} f(y,n-1) = f(x,n-1).$

Therefore, by (3) and (4),

(5) $\sum_{y=1}^{x} f(y,n-1) - \sum_{y=1}^{x-1} f(y,n-1) = f(x,n) - f(x-1,n).$

Now suppose, for arbitrary n and $x = k-1$,

(6) $\sum_{y=1}^{k-1} f(y,n-1) = f(k-1,n).$

Then, by (5),

$$\sum_{y=1}^{k} f(y,n-1) = f(k,n).$$

But (6) is true for $k = 2$ by (1). Therefore we have

$$\sum_{y=1}^{x} f(y,n-1) = f(x,n)$$

for all integral $x \geqq 1$. But, by (2), $f(x,1) = 1^2 + 2^2 + \cdots + x^2$. Thus the theorem is established.

Solved also by Leo Moser, Winnipeg, Canada.

No. 23. Proposed by V. Thebault, Tennie, Sarthe, France.

Given an orthocentric tetrahedron $ABCD$, of orthocenter H, show that the spheres (A), (B), (C), (D) of centers A, B, C, D, orthogonal to a sphere of center H, cut the planes of the faces BCD, CDA, DAB, ABC in four circles which lie on the same sphere.

Solution by C. W. Trigg, Los Angeles City College.

Let the altitude from A be AA', the sphere centered on H be (H) with radius R_h, the radius of (A) be R_a, and the radius of the circle C_a in which (A) intersects BCD be r_a. Then since AA' is perpendicular to BCD, A' is the center of C_a and $R_a^2 = r_a^2 + \overline{AA'}^2$. Since (A) and (H) are orthogonal, $R_a^2 + R_h^2 = \overline{AH}^2$. Similar relations hold for the other three circles. If C_a, C_b, C_c and C_d lie on a sphere (S) with radius R, the center of the sphere must lie on AA', BB', CC' and DD', so the center of (S) is H. Therefore, $R^2 = r_a^2 + \overline{HA'}^2 = \overline{AH}^2 - R_h^2 - \overline{AA'}^2 + \overline{HA'}^2 = \overline{AH}^2 + \overline{HA'}^2 - (\overline{AH} + \overline{HA'})^2 - R_h^2$ $= -2\,\overline{AH} \cdot \overline{HA'} - R_h^2$. But in an orthocentric tetrahedron the product of the segments into which the orthocenter divides the altitudes is constant, so $R^2 = -k - R_h^2$ will be obtained for the other three circles also. Hence C_a, C_b, C_c, C_d lie on a sphere (S) concentric with (H). In order that (S) may be real, $\overline{AH}$ and $\overline{HA'}$ must be oppositely directed, in which event H will fall outside the tetrahedron.

Solved also by Howard Eves, Corvallis, Oregon.

PROPOSALS

No. 24. Proposed by C. N. Mills, Normal, Illinois.

Given the quadratic form $X^2 - X + N$ where $N = 2, 3, 5, 11, 17$, and 41. When $X = 1, 2, 3, \cdots (N-1)$ each of the resulting numbers is prime. Are there other values of N?

No. 25. Proposed by K. E. Cappel, San Francisco, California.

Three points, A, B and C, are located on a straight line. A force f_1, of known magnitude and direction, passes through B. A second force f_2, coplanar with f_1 and ABC, and of known magnitude but unknown direction, is to pass through the point A in such a way that the resultant of f_1 and f_2 passes through the point C. Find the direction of f_2.

No. 26. Anonymous.

In certain research work in psychology, the following table is used:

$$\begin{array}{ccccc}
x_{11} & x_{12} & x_{13} & \cdots & x_{1r} \\
x_{21} & x_{22} & x_{23} & \cdots & x_{2r} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_{r1} & x_{r2} & x_{r3} & \cdots & x_{rr}
\end{array}$$

The sums of the products of the corresponding elements in every two rows are empirically determined, i.e. $\sum_{i=1}^{r} x_{mi}\, x_{ni} = C_{mn}$, $m,n \le r$, $m \ne n$. Discuss the possibility of finding the values of the x_{ij} for various values of r and secure a formula for these values of x_{ij} when they can be found.

Error — proposal 17, vol. XXII, no. 1, should read "Given an integer of n non-zero digits, show that it is alsays possible to replace a certain r $(0 \le r < n)$ of these digits by *zeros* in such a way that the resulting number is divisible by n."

MATHEMATICAL MISCELLANY

Edited by
Marian E. Stark

Let us know (briefly) of unusual and successful programs put on by your Mathematics Club, of new uses of mathematics, of famous problems solved, and so on. Brief letters concerning the MATHEMATICS MAGAZINE or concerning other "matters mathematical" will be welcome. Address: MARIAN E. STARK, Wellesley College, Wellesley 81, Mass.

We have received several poetical contributions. The following two we especially like.

How to Capitalize on Schizophrenia

My every impulse splits in two.
The halves then fight like fiend fanatics.
There's only one thing left to do:
Exploit the splits, in matrix mathematics.

Unfinished Symbol

All life is an infinite series of zeroes .....
With once in a while a significant figure.
But whether it's notable,
Or whether it's negligible,
Depends entirely on the location
Of the decimal point,
Death, so called.

H. W. Becker

An interesting book has come to hand by Pedro A. Pizá of San Juan, Puerto Rico. It is called *Arithmetical Essays,* "Numerical adventures of a devotee of arithmetic, undertaken in solitude for his own spiritual recreation."

Colonel Byrne sends us the following interesting news, received from Professor Julia in France:

"Professor Gaston Julia wrote 21 July 1948 about his trip to Geneva, where was held a Science Congress during the first part of July. The mathematical part was brought almost to the dimensions of a small international congress by the invitations extended by Professor Wavre of the University of Geneva. Besides professors of mathematics from Swiss universities there were Belgians (including de la Vallée-Poussin), one Hungarian (Fejer), two Bulgarians (Obrechkoff, Tschakuloff), two Poles (Sierpinski, Kuratowski). Among the Swiss present at the mathematical sessions sere: Ostrowski, Fueter, Hopf, de Rham, Wavre, Eckmann. Most of the discussions were about function theory, group theory and topology.

Professor Julia gave a lecture on 'Une généralisation des systèmes orthonormaux'. De Rham, Hopf, Kuratowski gave very interesting papers on

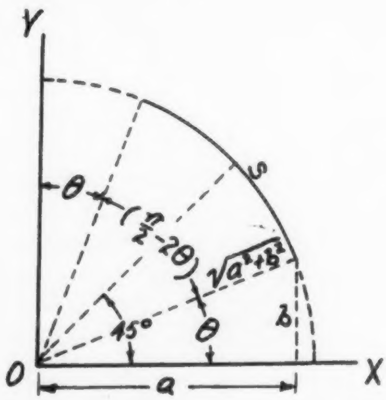
topology. Fejer lectured on singular integrals. Dieudonné gave an important lecture on the classical groups and their automorphisms."

Rigidity Restrictions on Analytic Loci

The following might properly be called a mathematical aside on "Mathematical Asides", to the clever paper which was delivered by Dr. N. A. Court of the University of Oklahoma to the St. Louis University Chapter of Pi Mu Epsilon on April 8, 1947 and published in Scripta Mathematica, Vol. XIII, March-June 1947.

In Dr. Court's paper the following classic problem was used as an example, which was most apropos (our quarrel, if it be one, is with the solution as usually given): A rigid ellipse moves so that it constantly remains tangent to the coordinate axes. Find the locus of the center of the ellipse.

The classic solution is a circle whose radius is equal to that of the director circle of the ellipse. Analytically the problem yields as the locus of the center the equation to such a circle. But granting that the given "rigid ellipse" can be moved from one quadrant to another it can generate only an arc of the same circle in each quadrant, and one wonders (1) what is the length of the arc generated in each quadrant? (2) After the first four arcs are generated through what angle must the fixed orthogonal axes be rotated about their intersection, and what must be the eccentricity of the ellipse so that when the four new arcs are now generated with axes in the new position the full locus circle is just completed without overlapping of arcs?



From the figure it is seen that $s = (a^2 + b^2)^{1/2}\ (\tfrac{1}{2}\pi - 2\tan^{-1} b/a) = R(\tfrac{1}{2}\pi - 2\theta)$ where a and b are as usual the semi-major and semi-minor axes of the ellipse. But $\tan\ (\tfrac{1}{2}\pi - 2\tan^{-1} b/a) = \cot\ (2\tan^{-1} b/a) = (a^2 - b^2)/2ab$ so that one may write $s = R\tan^{-1}\ (a^2 - b^2)/2ab = R\tan^{-1} D/H$, where H is the harmonic mean of a, b; D is the difference of a, b; R is the radius of the director circle of the ellipse.

From the figure it is seen that for the arcs to be generated without overlapping after rotation of the axes, the length of the arcs must be half the quadrant length and the angle of rotation must be 45°. Hence

$$\theta = \pi/8 = \tan^{-1} b/a, \quad \text{or} \quad b/a = \tan \pi/8 = 2^{1/2} - 1.$$

$$e^2 = 1 - b^2/a^2 = 1 - (2^{1/2} - 1)^2 = 2(2^{1/2} - 1).$$

Similarly the space analogue, i.e. the locus of the center of an ellipsoid which remains tangent to the three coordinate planes, is usually given as a sphere whose radius is equal to that of the director sphere of the given ellipsoid (radius squared = $a^2 + b^2 + c^2$, where a, b, c are the semi-axes of the ellipsoid). Again the true locus is only a portion of the spherical surface in one quadrant and one might ask similarly, what is the area of this generated portion? What conditions must be imposed on the ellipsoid and the coordinate system so that when the ellipsoid is allowed to generate the partial spherical surface in each of the eight quadrants we may generate the complete spherical locus without overlapping of areas? The reader may enjoy this investigation.

In a sense all this may seem trivial, but it is an example of which there are many similar ones where geometers lose sight of the limitations that the rigidity of the objects they are discussing put upon the purely geometrical results obtained.

Arlington, Va. Paul D. Thomas

Managing Editor
Mathematics Magazine
Pacoima, California

Dear Sir:

"The discovery that a cube root would act as an insecticide, is not properly understood, nor are decisions which uphold it sound, if we assume it is a discovery of principle of nature."

Such is the wording of the law as announced in the case of Dennis v. Pitner, Vol. 106 Federal Reporter, 2nd Series, beginning at page 142.

As a subscriber to your magazine, and a law student, I thought that the rather unusual wording of the quotation might be of interest to you.

Yours very truly,

Paul Kopp
